1. Consider the following differential equation:
\[ y''' - 6y'' + 9y' = x. \]
where \( y \) is a function of \( x \).

(a) (5 POINTS) Find the general solution to the corresponding homogeneous differential equation.

Solution: The auxiliary equation is
\[ 0 = r^3 - 6r^2 + 9r = r(r - 3)^2, \]
so the general solution is
\[ y = c_1 + (c_2 + c_3x)e^{3x}. \]

(b) (5 POINTS) Find a particular solution to the given nonhomogeneous equation.

Solution: If \( r = 0 \) were not a root of the auxiliary polynomial, we could use \( y = ax \) as a particular solution and solve for \( a \). In this case we have to use \( y = x(a_1x + a_0) \), which gives
\[ y''' - 6y'' + 9y' = 18a_1x + 9a_0 - 12a_1, \]
so \( a_1 = 1/18, a_0 = 2/27 \) and \( y_p = x(3x + 4)/54. \)
2. Convert each of the following differential equations to a system of first order differential equations.

(a) (5 points) \( y''' - 3y'' + 2y' - y = 0 \)

Solution: Let \( x_1 = y, \ x_2 = y', \) and \( x_3 = y''. \) Then we have

\[
\begin{align*}
  x_1' &= y' = x_2 \\
  x_2' &= y'' = x_3 \\
  x_3' &= y''' = 3y'' - 2y' + y \\
       &= 3x_3 - 2x_2 + x_1.
\end{align*}
\]

so the system is

\[
\begin{align*}
  x_1' &= x_2 \\
  x_2' &= x_3 \\
  x_3' &= 3x_3 - 2x_2 + x_1.
\end{align*}
\]

(b) (5 points) \( y'''' - 2y'' + y = 0 \)

Solution: Let \( x_1 = y, \ x_2 = y', \ x_3 = y'', \) and \( x_4 = y''''. \) Then we have

\[
\begin{align*}
  x_1' &= y' = x_2 \\
  x_2' &= y'' = x_3 \\
  x_3' &= y''' = x_4 \\
  x_4' &= y''' = 2y'' - y \\
       &= 2x_3 - x_1.
\end{align*}
\]

so the system is

\[
\begin{align*}
  x_1' &= x_2 \\
  x_2' &= x_3 \\
  x_3' &= x_4 \\
  x_4' &= 2x_3 - x_1.
\end{align*}
\]
3. (10 POINTS) Solve the following system of differential equations

\[
\begin{align*}
x_1' &= 4x_2 \\
x_2' &= x_1 
\end{align*}
\]

subject to the initial conditions

\[x_1(0) = 1 \quad \text{and} \quad x_2(0) = -2.\]

**Solution:** In matrix form this system is \(X' = AX\) for

\[
X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}
\]

The characteristic polynomial of \(A\) is \(P(\lambda) = \lambda^2 - 4\), so the eigenvalues are \(\lambda = \pm 2\). These yield eigenvectors \((2, i)\) and \((2, -1)\), so we have \(ST = AS\) where

\[
S = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.
\]

By letting \(X = SY\), we can rewrite the system as \(Y' = TY\), for which the general solution is

\[
\begin{align*}
y_1 &= c_1 e^{2t} \\
y_2 &= c_2 e^{-2t}
\end{align*}
\]

Since \(X = SY\), we get

\[
\begin{align*}
x_1 &= 2c_1 e^{2t} + 2c_2 e^{-2t} \\
x_2 &= c_1 e^{2t} - c_2 e^{-2t}.
\end{align*}
\]

The initial conditions give

\[
\begin{align*}
1 &= 2c_1 + 2c_2 \\
-2 &= c_1 - c_2.
\end{align*}
\]

so \(c_1 = -3/4\) and \(c_2 = 5/4\).
4. (10 POINTS) Determine the general solution to the following system of differential equations:

\[
\begin{bmatrix}
  x_1' \\
  x_2'
\end{bmatrix} = \begin{bmatrix}
  5 & -2 \\
  1 & 2
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
\]

**Solution:** The characteristic polynomial of the $2 \times 2$ matrix $A$ is

\[P(\lambda) = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4).\]

The eigenvalues 3 and 4 yield eigenvectors $(1, 1)$ and $(2, 1)$ respectively, so we get $T = S^{-1}AS$ where

\[S = \begin{bmatrix}
  1 & 2 \\
  1 & 1
\end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix}
  3 & 0 \\
  0 & 4
\end{bmatrix}.
\]

By letting $X = SY$, we can rewrite the system as $Y' = TY$, for which the general solution is

\[
\begin{align*}
  y_1 &= c_1 e^{3t} \\
  y_2 &= c_2 e^{4t}
\end{align*}
\]

Since $X = SY$, we get

\[
\begin{align*}
  x_1 &= c_1 e^{3t} + 2c_2 e^{4t} \\
  x_2 &= c_1 e^{3t} + c_2 e^{4t}.
\end{align*}
\]
5. (10 POINTS) Calculate the matrix exponential function $e^{At}$ for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$ 

**Solution:** It is easy to compute the powers of this matrix since $A^4 = I$. For any integer $n$ we have

$$A^{4n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^{4n+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$A^{4n+2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A^{4n+3} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It follows that the upper left entry of $e^{At}$ is

$$\sum_{n=0}^{\infty} \frac{t^{4n}}{(4n)!} - \sum_{n=0}^{\infty} \frac{t^{4n+2}}{(4n+2)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots = \cos t,$$

while the upper right entry is

$$\sum_{n=0}^{\infty} \frac{t^{4n+1}}{(4n+1)!} - \sum_{n=0}^{\infty} \frac{t^{4n+3}}{(4n+3)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \sin t.$$

The lower entries can be similarly calculated, and we get

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$
6. Let $A$ be the following $4 \times 4$ matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(a) (5 points) Determine the eigenvalues of $A$.

**Solution:** The characteristic polynomial is

$$P(\lambda) = \lambda^4 - 1 = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i),$$

so the eigenvalues are $\pm 1$ and $\pm i$.

(b) (5 points) Determine the eigenvectors corresponding to each of the eigenvalues of $A$.

**Solution:** The eigenvectors for $1$, $-1$, $i$ and $-i$ are respectively

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, \quad \text{and} \quad X_4 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}.$$

(c) (5 points) Does there exist an invertible matrix $S$ and a diagonal matrix $D$ such that $A = SDS^{-1}$? If yes, state what the matrices $S$ and $D$ are. If not, explain clearly why such matrices do not exist.

**Solution:** Since there are 4 linearly independent eigenvectors, the matrix is diagonalizable with

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$
7. (10 points) Let \( B \) be the following \( 3 \times 3 \) matrix:

\[
B = \begin{bmatrix}
4 & 0 & -2 \\
0 & 4 & 0 \\
1 & 0 & 1 \\
\end{bmatrix}
\]

Diagonalize \( B \) (i.e., find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( B = PDP^{-1} \)).

SOLUTION: The characteristic polynomial is

\[
P(\lambda) = (4 - \lambda)^2(1 - \lambda) + 2(4 - \lambda) \\
= (4 - \lambda)((4 - \lambda)(1 - \lambda) + 2) \\
= (4 - \lambda)(4 - 5\lambda + \lambda^2 + 2) \\
= (4 - \lambda)(3 - \lambda)(2 - \lambda),
\]

so the eigenvalues are 2, 3 and 4. To find the eigenvectors, we have

\[
A - 2I = \begin{bmatrix}
2 & 0 & -2 \\
0 & 2 & 0 \\
1 & 0 & -1 \\
\end{bmatrix} \quad \text{so} \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\]

\[
A - 3I = \begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & 0 \\
1 & 0 & -2 \\
\end{bmatrix} \quad \text{so} \quad X_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}
\]

\[
A - 4I = \begin{bmatrix}
0 & 0 & -2 \\
0 & 0 & 0 \\
1 & 0 & -3 \\
\end{bmatrix} \quad \text{so} \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

so

\[
P = \begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix}
2 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 4 \\
\end{bmatrix}.
\]
8. (10 points) Find an orthonormal basis for the subspace of \( \mathbb{R}^5 \) spanned by the following vectors:

\[
(2, 0, 2, 0, 1), \ (2, 3, 2, 4, 1), \text{ and } (2, -1, 2, 7, 1)
\]

**Solution:** Call the vectors \( v_1, v_2 \) and \( v_3 \). We first use the Gram-Schmidt procedure to find an orthogonal basis, which we will normalize later. We have

\[
\begin{align*}
u_1 &= v_1 = (2, 0, 2, 0, 1) \\
u_2 &= v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 \\
&= (2, 3, 2, 4, 1) - \frac{(4 + 4 + 1)}{(4 + 4 + 1)} (2, 0, 2, 0, 1) \\
&= (0, 3, 0, 4, 0) \\
u_3 &= v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} u_2 \\
&= (2, -1, 2, 7, 1) - \frac{(4 + 4 + 1)}{(4 + 4 + 1)} (2, 0, 2, 0, 1) - \frac{(-3 + 28)}{(9 + 16)} (0, 3, 0, 4, 0) \\
&= (0, -4, 0, 3, 0).
\end{align*}
\]

It follows that the orthonormal basis is

\[
\left\{ \frac{1}{3} (2, 0, 2, 0, 1), \frac{1}{5} (0, 3, 0, 4, 0), \frac{1}{5} (0, -4, 0, 3, 0) \right\}.
\]

9. (10 points) Find the general solution for the following differential equation:

\[y''' - 2y'' + y = 0\]

**Solution:** The auxiliary polynomial is

\[r^4 - 2r^2 + 1 = (r^2 - 1)^2 = (r - 1)^2 (r + 1)^2,\]

so the general solution is

\[y = (c_1 + xc_2) e^x + (c_3 + xc_4) e^{-x}.\]