Part I

1. (10 points) Solve the initial value problem \( y'' - 3y' + 2y = 0 \) with \( y(0) = 1 \) and \( y'(0) = -1 \).

   **Solution:** The auxiliary polynomial is
   \[ r^2 - 3r + 2 = (r - 1)(r - 2), \]
   so the general solution is
   \[ y = c_1 e^x + c_2 e^{2x}. \]
   The initial conditions give
   \[ \begin{align*}
   c_1 + c_2 &= 1 \\
   c_1 + 2c_2 &= -1,
   \end{align*} \]
   which gives \( c_1 = 3 \) and \( c_2 = -2 \), so the solution is
   \[ y = 3e^x - 2e^{2x}. \]

2. (10 points) Find the general solution to \( y'' - 3y' - 4y = e^x \).

   **Solution:** The auxiliary polynomial is
   \[ r^2 - 3r - 4 = (r + 1)(r - 4), \]
   so the complimentary solution is
   \[ y = c_1 e^{-x} + c_2 e^{4x}. \]
Since $e^x$ does not satisfy the homogeneous equation, we can use $y = ae^x$ as our particular solution and solve for the undetermined coefficient $a$. We have

$$y'' - 3y' - 4y = (1 - 3 - 4)ae^x = -6ae^x = e^x$$

so $a = -1/6$ and the general solution is

$$y = c_1e^{-x} + c_2e^{4x} - e^x/6.$$ 

3. (10 POINTS) Let

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Find numbers $a$, $b$ and $c$ so that $M^3 + aM^2 + bM + cI_3 = 0$, where $I_3$ is the $3 \times 3$ identity matrix.

**Solution:** We have

$$M^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M^3 = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

so

$$M^3 + aM^2 + bM + cI_3 = \begin{bmatrix} 1 + a + b + c & 3 + 2a + b & 6 + 3a + b \\ 0 & 1 + a + b + c & 3 + 2a + b \\ 0 & 0 & 1 + a + b + c \end{bmatrix}$$

and we have to solve the system

$$a + b + c = -1$$
$$2a + b = -3$$
$$3a + b = -6,$$

which gives $a = -3$, $b = 3$ and $c = -1$, so

$$M^3 - 3M^2 + 3M - I_3 = 0.$$
4. (10 points) Find the inverse of the matrix in Problem 3.

**SOLUTION:** We can do this with elementary row operations

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

so

\[M^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.\]

5. (10 points) Find the determinant of the matrix

\[M = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 5 & 0 & 6 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & 7 & 0 & 8 \end{bmatrix}.\]

**SOLUTION:** Using elementary row operations, we have

\[
\begin{vmatrix}
1 & 0 & 2 & 0 \\
5 & 0 & 6 & 0 \\
0 & 3 & 0 & 4 \\
0 & 7 & 0 & 8 \\
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 2 & 0 \\
0 & 0 & -4 & 0 \\
0 & 3 & 0 & 4 \\
0 & 0 & 0 & -4/3 \\
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & -4 & 0 \\
0 & 0 & 0 & -4/3 \\
\end{vmatrix}
= -16.
\]

3
6. (10 POINTS) Solve the system

\[
\begin{align*}
    x_1 + 2x_3 &= 1 \\
    3x_2 + 4x_4 &= -1 \\
    5x_1 + 6x_3 &= 1 \\
    7x_2 + 8x_4 &= -1.
\end{align*}
\]

**Solution:** We do elementary row operations on the augmented matrix. We have

\[
\begin{pmatrix}
    1 & 0 & 2 & 0 & 1 \\
    0 & 3 & 0 & 4 & -1 \\
    5 & 0 & 6 & 0 & 1 \\
    0 & 7 & 0 & 8 & -1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & 0 & 2 & 0 & 1 \\
    0 & 3 & 0 & 4 & -1 \\
    0 & 0 & -4 & 0 & -4 \\
    0 & 0 & 0 & -4/3 & 4/3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & 0 & 0 & 0 & -1 \\
    0 & 1 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0 & 1 \\
    0 & 0 & 0 & 1 & -1
\end{pmatrix}
\]

so the solution is

\[
\begin{align*}
    x_1 &= -1 \\
    x_2 &= 1 \\
    x_3 &= 1 \\
    x_4 &= -1.
\end{align*}
\]
7. (10 POINTS) Solve the system

\[
\begin{align*}
x_1 + 2x_3 &= 3 \\
3x_2 + 4x_4 &= 4 \\
5x_1 + 6x_3 &= 5.
\end{align*}
\]

**Solution:** Proceeding as in the previous problem we have

\[
\begin{bmatrix}
1 & 0 & 2 & 0 & 3 \\
0 & 3 & 0 & 4 & 4 \\
5 & 0 & 6 & 0 & 5
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & 2 & 0 & 3 \\
0 & 1 & 0 & 4/3 & 4/3 \\
0 & 0 & 1 & 0 & 5/2
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & -2 \\
0 & 1 & 0 & 4/3 & 4/3 \\
0 & 0 & 1 & 0 & 5/2
\end{bmatrix}
\]

so the solution is

\[
\begin{align*}
x_1 &= -2 \\
x_2 &= 4/3(1 - x_4) \\
x_3 &= 5/2.
\end{align*}
\]

Part II

8. Let \( A \) be the following \( 3 \times 3 \) matrix:

\[
A = \begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
2 & -1 & 1
\end{bmatrix}
\]

(a) (5 POINTS) Determine the eigenvalues of \( A \).

**Solution:** The characteristic polynomial is \((2 - \lambda)(1 - \lambda)^2\), so the eigenvalues are 1 and 2.

(b) (5 POINTS) Determine the eigenvectors corresponding to each of the eigenvalues of \( A \).
Solution: For $\lambda = 1$ we have
\[
A - I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\]
so an eigenvector is $(0, 0, 1)$. For $\lambda = 2$ we have
\[
A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{bmatrix}
\]
so an eigenvector is $(1, 0, 2)$.

(c) (5 points) Does there exist an invertible matrix $S$ and a diagonal matrix $D$ such that $A = SDS^{-1}$? If yes, state what the matrices $S$ and $D$ are. If not, explain clearly why such matrices do not exist.

Solution: There is no such $D$ or $S$ because the matrix $A$ is defective; it only has two linearly independent eigenvectors.

9. (10 points) Let $B$ be the following $2 \times 2$ matrix:
\[
B = \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}
\]
Diagonalize $B$ (i.e., find an invertible matrix $P$ and a diagonal matrix $D$ such that $B = PDP^{-1}$).

Solution: The characteristic polynomial is $(\lambda - 3)(\lambda + 2)$. For $\lambda = 3$ we have
\[
A - 3I = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}
\]
so an eigenvector is $(1, -2)$. For $\lambda = -2$ we have
\[
A + 2I = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}
\]
so an eigenvector is $(2, 1)$. It follows that

\[
P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.
\]

10. (10 points) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the following vectors:

$(1, 0, -1, 0), (1, 1, -1, 0), \text{ and } (-1, 1, 0, 1)$

**Solution:** Let $v_1, v_2, \text{ and } v_3$ be the given three vectors. We shall determine an ONB $\{u_1, u_2, u_3\}$ of the given subspace by the Gram-Schmidt process.

Define $u_1$ by rescaling $v_1$ to be a unit vector:

\[
u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} (1, 0, -1, 0)
\]

Define $w_2$ as follows:

\[
w_2 = v_2 - \langle v_2, u_1 \rangle u_1
\]

\[= (1, 1, -1, 0) - (1, 0, -1, 0)
\]

\[= (0, 1, 0, 0)
\]

Define $u_2$ by rescaling $w_2$ to be a unit vector:

\[
u_2 = \frac{w_2}{\|w_2\|} = (0, 1, 0, 0)
\]

Define $w_3$ as follows:

\[
w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2
\]

\[= (-1, 1, 0, 1) + \frac{1}{2} (1, 0, -1, 0) - 1(0, 1, 0, 0)
\]

\[= -\frac{1}{2} (1, 0, 1, -2)
\]

Define $u_3$ by rescaling $w_3$ to be a unit vector:

\[
u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{6}} (1, 0, 1, -2)
\]
An ONB for the subspace spanned by \( \{v_1, v_2, v_3\} \) is given by
\[
\left\{ \frac{1}{\sqrt{2}}(1, 0, -1, 0), \quad (0, 1, 0, 0), \quad \frac{1}{\sqrt{6}}(1, 0, 1, -2) \right\}
\]

11. (10 POINTS) Find the general solution for the following differential equation:
\[
y''' - y'' + y' - y = 0
\]

**SOLUTION:** The auxiliary equation corresponding to this DE is
\[
r^3 - r^2 + r - 1 = 0
\]
Inspection shows that \( r = 1 \) is a root of this equation. By long division we find that the polynomial factors as \( (r - 1)(r^2 + 1) \). The roots are 1 and \( \pm i \).
Therefore, the general solution to the DE is
\[
y = c_1 e^t + c_2 \cos(t) + c_3 \sin(t)
\]

12. Consider the following differential equation:
\[
y''' - 4y'' + 4y' = 4.
\]
(a) (5 POINTS) Find the general solution to the corresponding homogeneous differential equation.

**SOLUTION:** The auxiliary equation corresponding to the given DE is
\[
r^3 - 4r^2 + 4r = 0,
\]
which factors as \( r(r - 2)^2 \). Therefore, the general solution to the corresponding homogeneous equation is
\[
y_0 = c_1 + c_2 e^{2t} + c_3 t e^{2t}
\]
(b) (5 POINTS) Find a particular solution to the given nonhomogeneous equation.

**SOLUTION:** By the method of undetermined coefficients, we would look for a solution \( y_p \) of the form \( A \) where \( A \) is a constant. But the constant function is already a solution of the corresponding homogeneous equation. Therefore, we modify \( y_p \) to be
\[
y_p = At
\]
Taking the derivatives of $y_p$ and substituting into the nonhomogeneous equation, we get

$$4A = 4$$

and hence $A = 1$.

Therefore, the general solution to the given nonhomogeneous equation is

$$y = y_0 + y_p$$
$$= c_1 + c_2e^{2t} + c_3te^{2t} + t$$

13. (10 POINTS) Solve the following system of differential equations

$$x_1' = 2x_2$$
$$x_2' = -2x_1$$

subject to the initial conditions

$$x_1(0) = 2 \quad \text{and} \quad x_2(0) = -3.$$ 

**SOLUTION:** Rewrite the system in operator notation as

$$Dx_1 - 2x_2 = 0$$
$$2x_1 + Dx_2 = 0$$

Adding $D$ times the first equation to twice the second equation give the following DE for the function $x_1$:

$$(D^2 + 4)x_1 = 0$$

The auxiliary equation for this DE is $r^2 + 4 = 0$, which has two complex roots, $\pm 2i$. Therefore, the general solution for $x_1$ is

$$x_1(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

Substituting for $x_1$ in the first equation of the original system, we solve for $x_2$:

$$x_2 = \frac{1}{2}x_1' = -c_1 \sin(2t) + c_2 \cos(2t)$$
Using the initial conditions, we find that $c_1 = 2$ and $c_2 = -3$. Therefore, the solution of the original system is
\[
\begin{pmatrix}
  x_1(t) \\
  x_2(t)
\end{pmatrix}
= \begin{pmatrix}
  2 \cos(2t) - 3 \sin(2t) \\
  -2 \sin(2t) - 3 \cos(2t)
\end{pmatrix}.
\]

Alternatively, we can solve the system by diagonalizing the matrix
\[
A = \begin{bmatrix}
  0 & 2 \\
  -2 & 0
\end{bmatrix}.
\]
Its eigenvalues are $\pm 2i$, and its eigenvectors are $(2, i)$ (for $\lambda = i$) and $(2, -i)$. Thus we have $AS = SD$ where
\[
S = \begin{bmatrix}
  2 & 2 \\
  i & -i
\end{bmatrix}
\quad \text{and} \quad
D = \begin{bmatrix}
  2i & 0 \\
  0 & -2i
\end{bmatrix}.
\]
We let $X = SY$, and rewrite the system as $Y' = DY$, which gives
\[
y_1 = c_1 e^{2it} \\
y_2 = c_2 e^{-2it} \\
x_1 = 2y_1 + 2y_2 = 2c_1 e^{2it} + 2c_2 e^{-2it} \\
x_2 = iy_1 - iy_2 = ic_1 e^{2it} - ic_2 e^{-2it}.
\]
(These constants are not the same as the ones above.) The initial conditions imply
\[
2 = 2c_1 + 2c_2 \\
-3 = ic_1 - ic_2 \\
c_1 = (1 + 3i)/2 \\
c_2 = (1 - 3i)/2 \\
x_1 = (1 + 3i) e^{2it} + (1 - 3i) e^{-2it} \\
= \cos(2t) - 3 \sin(2t) \\
x_2 = (-3 + i) e^{2it} / 2 + (-3 - i) e^{-2it} / 2 \\
= -2 \sin(2t) - 3 \cos(2t).
\]
14. (10 POINTS) Determine the general solution to the following system of differential equations:

\[
\begin{bmatrix}
    x_1' \\
    x_2'
\end{bmatrix} =
\begin{bmatrix}
    -2 & -7 \\
    -1 & 4
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
\]

**Solution:** We first rewrite the system in operator notation as follows:

\[
(D + 2)x_1 + 7x_2 = 0 \\
x_1 + (D - 4)x_2 = 0
\]

Subtracting the first equation from \((D+2)\) times the second equation leads to the following DE for the function \(x_2\):

\[
((D + 2)(D - 4) - 7)x_2 = 0
\]

This can be simplified as

\[
x_2'' - 2x_2' - 15x_2 = 0
\]

The auxiliary equation corresponding to the above DE is

\[
r^2 - 2r - 15 = 0
\]

which has roots \(r_1 = 5\) and \(r_2 = -3\). Therefore, the general solution for \(x_2\) is

\[
x_2(t) = c_1 e^{5t} + c_2 e^{-3t}
\]

Substituting this into the second equation in the original system and solving for \(x_1\), we get

\[
x_1 = 4x_2 - x_2' = 4(c_1 e^{5t} + c_2 e^{-3t}) - (5c_1 e^{5t} - 3c_2 e^{-3t}) = -c_1 e^{5t} + 7c_2 e^{-3t}
\]

We can write the solution to the original system in vector form as follows:

\[
\begin{pmatrix}
    x_1(t) \\
    x_2(t)
\end{pmatrix} = c_1 e^{5t} \begin{pmatrix}
    -1 \\
    1
\end{pmatrix} + c_2 e^{-3t} \begin{pmatrix}
    7 \\
    1
\end{pmatrix}
\]
Alternatively, we can solve the system by diagonalizing the matrix

\[ A = \begin{bmatrix} -2 & -7 \\ -1 & 4 \end{bmatrix}. \]

Its characteristic polynomial is

\[ (-2 - \lambda)(4 - \lambda) - 7 = \lambda^2 - 2\lambda - 8 - 7 = (\lambda - 5)(\lambda + 3). \]

Its eigenvalues are 5 and -3, and its eigenvectors are (-1, 1) and (7, 1). Thus we have \( AS = SD \) where

\[ S = \begin{bmatrix} -1 & 7 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}. \]

We let \( X = SY \), and rewrite the system as \( Y' = DY \), which gives

\[
\begin{align*}
y_1 & = c_1e^{5t} \\
y_2 & = c_2e^{-3t} \\
x_1 & = -y_1 + 7y_2 \\
& = -c_1e^{5t} + 7c_2e^{-3t} \\
x_2 & = y_1 + y_2 \\
& = c_1e^{5t} + c_2e^{-3t}.
\end{align*}
\]