Math 236H Second midterm exam

April 3, 2008

Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

1. (20 POINTS) You are painting cubic blocks and you have n colors to choose from for each of the 6 faces. How many distinguishable blocks can be made in this way? How big does n have to be in order to get at least 50 distinguishable blocks?

Solution: We will use Burnside's formula

$$r = \sum_{g \in G} |X_g|/|G|$$

to find the number of orbits r. The group G here is the symmetry group of the cube, which is isomorphic to S_4, and the set X has n^6 elements. Conjugate lements in the group will have fixed points sets of the same size, and there are five such classes:

- The identity element, which fixes all n^6 colorings.
- The six 4-cycles, each of which corresponds to a $\pi/2$ (90 degrees) rotation about the center of a face. If the top face is being rotated, the 4 vertical faces must all have the same color, while the top and bottom faces can be coloered independently. This means there are n^3 colorings that are fixed.
- The three double transpositions, which are squares of 4-cylces. Each corresponds to π (180 degrees) rotation about the center of a face. here the front and back faces must have the same color, as shold the left and right faces. There are n^4 colors that are fixed.
- The eight 3-cycles. Each correpsonds to a $2\pi/3$ (120 degrees) rotation about a vertex. The three faces surrounding that vertex msut have the same color, as should the other three faces, so n^2 colorings are fixed.
- The six transpositions. Each corresponds to a π (180 degrees) rotation about the center of an edge. A coloring is fixed by the rotation of the two faces adjoining the edge have the same color, the two adjoining the opposite edge have a second color, and the two remaining faces (which also get swapped) have a third color. Hence n^3 colorings are fixed.

It follows that Burnside's sum is

$$\begin{aligned} r &= \frac{n^6 + 6n^3 + 3n^4 + 8n^2 + 6n^3}{24} \\ &= \frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24} \end{aligned}$$

This number is 10 for $n = 2$ and 57 for $n = 3$.

2. (10 POINTS) Let X be the set of ordered pairs (i, j) where i and j are integers ranging from 1 to 5. Let the symmetric group S_5 act on this set by permuting the integers in the usual way. Describe the orbits of this S_5-set. You do not need Burnside's formula for this.

Solution: There are two orbits: One consists of the 20 ordered pairs (i, j) with $i \neq j$, and the other consists of the five pairs (i, i).

3. (20 POINTS) Find the center $Z(G)$ and the commutator subgroup $C(G)$ for the group $G = C_{11} \times S_3$.

Solution: We first prove that the center/commutator subgroup of a direct product $G_1 \times G_2$ of two groups is the product of the centers/commutator subgroups.

- CENTERS: $(g_1, g_2) \in G_1 \times G_2$ commutes with every element iff g_1 commutes with each element in G_1 and g_2 commutes with each element in G_2.
- COMMUTATOR SUBGROUPS:

$$\begin{aligned}
[(g_1, g_2)(h_1, h_2)] &= (g_1, g_2)(h_1, h_2)(g_1, g_2)^{-1}(h_1, h_2)^{-1} \\
&= (g_1, g_2)(h_1, h_2)(g_1^{-1}, g_2^{-1})(h_1^{-1}, h_2^{-1}) \\
&= (g_1 h_1 g_1^{-1} h_1^{-1}, g_2 h_2 g_2^{-1} h_2^{-1}) \\
&= ([g_1, h_1], [g_2, h_2]) \\
&= ([g_1, h_1], e_{G_2})(e_{G_1}, [g_2, h_2])
\end{aligned}$$

  so each commutator in $G_1 \times G_2$ is the product of one in G_1 with one in G_2.

Now the centers are $Z(C_{11}) = C_{11}$ and $C(S_3) = \{e\}$, so $Z(C_{11} \times S_3) = C_{11}$.
The commutator subgroups are $C(C_{11}) = \{e\}$ and $C(S_3) = C_3$, so $C(C_{11} \times S_3) = C_3$.

4. (10 POINTS) Describe the center of every simple

   a. abelian group
   b. nonabelian group

Solution: The center of any group is normal. By defintion a simple group G has only two normal subgroups, itself and $\{e\}$.

a. For any abelian group A, including a simple one, the center is A.

b. The center is abelian, so it cannot by G, so it must be $\{e\}$.

5. (20 POINTS) Let H be a normal subgroup of G of index m. Show that $g^m \in H$ for every $g \in G$.

Solution: The quotient group G/H has order m, which means the mth power of each element in it is the identity. Thus g^m is in the kernel of the homomorphism $G \to G/H$, which is H.