Recall if a finite group $G$ acts on a finite set $X$, then the number of orbits is

$$\lambda = \frac{1}{|G|} \sum_{g \in G} |X_g|$$

Burnside's Theorem

$X_g = \text{subset of } X \text{ fixed by } g$

$|X_g| = \# \text{ of elements in } |X_g|$

Claim: $g' = \text{high } \implies \text{ i.e. } g'$ is
conjugate to $g$) then $|X_g'| = |X_g|$

Proof: Let $x \in X_g$, i.e. $g(x) = x$

Then $g'(h(x)) = h \cdot g \cdot h^{-1} \cdot h(x)$

$= h \cdot g(x) = h(x)$

so $h(x) \in X_{g'}$. Similarly if $y \in X_{g'}$ then $h^{-1}(y) \in X_g$.

We get a 1-1 correspondence.
between $X_g$ and $X_{g'}$. QED

Example 0. Consider triangles with colored edges. There are $n$ colors. $C_3$ acts on a triangle by rotations and reflections.
There $n^3$ ways to paint a triangle. How many orbits

- $n$ triangles with one color
- $n(n-1)$
- $\frac{n(n-1)(n-2)}{6} = \binom{n}{3}$

1. two
2. three
Total # of outcomes = \( n + n^2 - n + \binom{n}{2} \)

\[
= n^2 + \frac{n^3 - 3n^2 + 2n}{6} = \frac{n^3 + 3n^2 + 2n}{6}
\]
We will use Burnside's formula to get this.

In $G_1 = S_3$, we have

$e = \text{identity element}$

three reflections $M$

two rotations $P$

$|X| = n^3$
\[ |X_e| = n^3 \quad 2 |X_0| = n \]

\[ |X_m| = n^2 \]

Burnside's Lemma:

\[ = n^3 + 3n^2 + 2n \]

\# of orbits = \[ \frac{n^3 + 3n^2 + 2n}{6} \]
(2) Tetrahedral blocks.

Each of 4 faces is painted one of \( n \) colors.

\[ |X| = n^4 \]

\( G = A_4 \) order 12

one identity element e
notations of order 2, $p_2$,

$$|X_e| = n^4$$

Burnside's sum

$$|X_{p_2}| = n^2$$

$$|X_{p_3}| = n^2$$

# of orbits $\approx \frac{n^4 + 11n^2}{12}$
\[
\begin{align*}
\text{n} & \quad \left( n^4 + 11n^2 \right) / 12 \\
\text{1} & \quad \left( 1^4 + 11 \cdot 1^2 \right) = 12 \\
\text{2} & \quad \left( 16 + 44 \right) / 12 = 5 \\
\text{3} & \quad \left( 81 + 99 \right) / 12 = 15 \\
\text{4} & \quad \left( 256 + 176 \right) / 12 = 36 \\
\text{5} & \quad \left( 625 + 275 \right) / 12 = 75
\end{align*}
\]
3. Cubic blocks

\[ \text{lxl} = n^6 \quad G = S_4 \]
Vertices of cube = \((\pm1, \pm1, \pm1)\)

A vertex is green if \(xyz = 1\)

A vertex is blue if \(xyz = -1\)

In \(S_4\) we have

- identity element \(e\)
- \(3\) double transpositions
- \(3\) \(S_3\) elements of order 3
- \(6\) rotations of order 4
notations about edge centers

$|X_e| = n^6$

$|X_{pq}| = n^3$

$|X_{e3}| = n^2$

$|X_{e5}| = n$

Burnside's lemma

$= n^6 + 6n^2 + 3n^4 + 12n^3$
# of orbits

\[ = n^6 + 3n^4 + 12n^3 + 8n^2 \]

\[ \frac{24}{n} \]

\[ n^6 + 3n^4 + 12n^3 + 8n^2 \]

\[ \frac{24}{24} \]

\[ 1 \]

\[ \frac{64 + 48 + 96 + 32}{24} \]

\[ = 10 \]
3 \left| \begin{array}{c}
729 + 243 + 324 + 72 \\
24
\end{array} \right.
= 57

Consider the polynomial \( (n)_k \) (\( k \) fixed)

\[ (n)_k = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} \]
\( (n) = \frac{n^2 - 3n + 2}{6} \)

There are numerical polynomials, i.e., polynomials in \( \mathbb{Z} \) with rational coefficients, s.t. \( n \in \mathbb{Z} \) then \( f(n) \in \mathbb{Z} \). Thus any numerical...
polynomial is an integer linear combination of these.