Lemma 37.5 Let $H$ and $K$ be normal subgroups of $G$ such
1) they generate all of $G$
2) $H \cap K = \{e\}$

Then $G \cong H \times K$

Proof. Will show that each $h \in H$
commutes with each $k \in K$. Consider
their commutation

$[h, k] = hkh^{-1}k^{-1}$

$= (hk(h^{-1}k^{-1}))k^{-1} = h(kh^{-1}k^{-1})$
Hence \([h, k] \in H \cap K = e \times e\)
\[h k h^{-1} k^{-1} = e\]
\[h k h^{-1} k^{-1} = k^r\]  
Smlt both sides on right by \(k\)

\[h k = k h\]

\(h\) and \(k\) commute.

We get a \(H \times K \rightarrow G\) which in iso

Thm 3.7.b Suppose \(|G| = p^r, p\) prime
Then \(G\) is abelian

\[m_{K} \text{ because } K \text{ is normal}\]
Proof: We know $C_p^2$ and $C_p \times C_p$ have order $p^2$. Suppose $G$ is not cyclic.

Pick $a, b \in G$, $a, b \neq 1$ with $b \notin \langle a \rangle$ and $a \notin \langle b \rangle$.

By 36.8, each subgroup $\langle a \rangle$ and $\langle b \rangle$ is normal and have order $p$. Previous Lemma shows $G \cong C_p \times C_p$.

Q.E.D.

Thm 37.7: Let $p < q$ be primes with $161 = pq$. If $q \neq 1 \mod p$
then \( G \cong C_{pq} \).

**Proof.** \( G \) has a \( p \)-Sylow \( H \subset G \).

By Sylow 3, the \# of such subgroups must divide \( p^q \), i.e. be 1, \( p \), \( q \), \( pq \) and is 1 \( \mod p \), so it is 1.

Similarly there is only one subgroup of order \( q \). Both subgroups normal \( \Rightarrow \) \( G \cong C_p \times C_q \cong C_{pq} \), Q.E.D.
Review

Sylow Thms: \( G \) finite \( \neq \)

\( p \) a prime \( |G| = m p^n \) with \( p \nmid m \).

1. \( G \) has subgroups \( \{e^g \} = H_0 \subset H_1 \subset H_2 \cdots \subset H_m \subset G \) where \( |H_i| = p^i \) and \( H_i \) is normal in \( H_{i+1} \).

\( H = H_m \) is called a \( p \)-Sylow subgroup of \( G \).

2. Any 2 such subgroups are conjugate and hence isomorphic.

3. The \( \# \) \( r \) of such subgroups
divides $|G|$ and $k = 1 \mod p$. 

#2 on practice exam

$S_4$ acts on $\leq (i, j): 1 \leq i, j \leq 4$ by permuting integers 1 through 4. Describe the orbits.

$(2, 3)$ and $(1, 4)$ are in the same orbit. $(1, 3)$ and $(1, 1)$ not in the same orbit.
There are 2 outputs ??

a) Four sets of the form \((x, x)\)

b) 12 sets \((x, y)\) with \(x \neq y\)

\[
x = \sum \left( i, j, k \right) \; 1 \leq i, j, k \leq 4 \bigg/ \left( x, x, x \right)
\]

\[
\sum \left( x, y, z \right) \; y \neq y \bigg/ \left( x, x, y \right)
\]
If \( (x, y, z) : x \neq y \) then
\[
\begin{align*}
\exists (y, x, z) : x &= y \\
\exists (x, y, z) : x &= y \\
\exists (x, y, z) : x, y \neq z \quad \text{distinct} \\
\end{align*}
\]

12
12
24

# Octahedral blocks

In colors

\(|X| = n^8 \quad G \subset S_4\)
Burnside's lemma

\[ \sum_{\text{\text{conj classes}}} |X^g| \]

Conjugacy classes in \( S_n \)

1. \( \bar{g} \neq g \)
2. \( \bar{g} \notin \{(gB)\} \)

rotate about line
b \rightarrow d \quad a \rightarrow h

e \rightarrow g \quad c \rightarrow f

3 \{ (a \land b), (g \land h) \}

\text{Mirror} \ 180^\circ
Barnes' sum

\[ n^8 + 6n^2 + 17n^4 + \frac{1}{4} \]

# of orbits is

\[ n^8 + 17n^4 + 6n^2 - 24 \]

\[
\begin{array}{c|c|c|c|c|c|c}
 n & \text{sum} & & & & & \\
1 & 24 & & & & \frac{6561}{54} & 1377 \\
2 & 256 + 24 + 272 = 552 = 24 \cdot 23 & & & & \frac{1377}{7992} & 24 \cdot 333 \\
3 & 6561 + 54 + 1377 = 7992 = 24 \cdot 333 & & & & & \\
\end{array}
\]