1. (20 points) You are painting cubic blocks and you have \( n \) colors to choose from for each of the 6 faces. How many distinguishable blocks can be made in this way? How big does \( n \) have to be in order to get at least 100 distinguishable blocks?

**Solution:** We will use Burnside’s formula

\[
r = \sum_{g \in G} |X_g|/|G|
\]

to find the number of orbits \( r \). The group \( G \) here is the symmetry group of the cube, which is isomorphic to \( S_4 \), and the set \( X \) has \( n^6 \) elements. Conjugate elements in the group will have fixed points sets of the same size, and there are five such classes:

- The identity element, which fixes all \( n^6 \) colorings.
- The six 4-cycles, each of which corresponds to a \( \pi/2 \) (90 degrees) rotation about the center of a face. If the top face is being rotated, the 4 vertical faces must all have the same color, while the top and bottom faces can be colored independently. This means there are \( n^3 \) colorings that are fixed.
- The three double transpositions, which are squares of 4-cycles. Each corresponds to \( \pi \) (180 degrees) rotation about the center of a face. Here the front and back faces must have the same color, as should the left and right faces. There are \( n^4 \) colors that are fixed.
- The eight 3-cycles. Each corresponds to a \( 2\pi/3 \) (120 degrees) rotation about a vertex. The three faces surrounding that vertex must have the same color, as should the other three faces, so \( n^2 \) colorings are fixed.
- The six transpositions. Each corresponds to a \( \pi \) (180 degrees) rotation about the center of an edge. A coloring is fixed by the rotation of the two faces adjoining the edge have the same color, the two adjoining the opposite edge have a second color, and the two remaining faces (which also get swapped) have a third color. Hence \( n^3 \) colorings are fixed.

It follows that Burnside’s sum is

\[
r = \frac{n^6 + 6n^3 + 3n^4 + 8n^2 + 6n^3}{24} = \frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}
\]

This number is 57 for \( n = 3 \) and 240 for \( n = 4 \).
2. (20 points) Determine the number of subgroups of order 3 in

a. $C_{27} \times C_{27}$

b. $C_9 \times C_9 \times C_9$

c. $C_3 \times C_3 \times C_3 \times C_3$

Solution:

a. Any subgroup of order 3 must be contained in $C_3 \times C_3$. It has 8 nontrivial elements, and each subgroup of order three contains two of them, so there are four such subgroups.

b. Any subgroup of order 3 must be contained in $C_3 \times C_3 \times C_3$, which has 26 nontrivial elements, so there are 13 subgroups of order 3.

c. $C_3 \times C_3 \times C_3 \times C_3$ also has 80 nontrivial elements and 40 subgroups of order 3.

3. (20 points) Describe a nonabelian group of order 57. Hint: Consider the group of $2 \times 2$ matrices (under multiplication) of integers modulo 19 of the form

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

with $a \neq 0$.

Solution: The matrix group $G$ is nonabelian since

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} aa' & ab' + b \\ 0 & 1 \end{bmatrix}$$

while

$$\begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a'a & ab+b' \\ 0 & 1 \end{bmatrix}$$

and $ab' + b \neq a'b + b'$ in general. It has order 342 since there are 18 possible values of $a$ and 19 possible values of $b$. It has a normal subgroup $B$ of order 19 consisting of matrices with $a = 1$. $G/B$ is isomorphic to the subgroup of order 18 consisting of matrices with $b = 0$, which is abelian and cyclic. $G/B$ has a subgroup of order 3 in which $a \in \{1, 7, 11\}$; note that $7^3$ is congruent to 1 modulo 19. Hence our group of order 57 is the set of matrices as above with $a \in \{1, 7, 11\}$.

4. (20 points) Let $X$ be the set of ordered triples $(i, j, k)$ where $i$ and $j$ are integers ranging from 1 to 4. Let the symmetric group $S_4$ act on this set by permuting the integers in the usual way. Describe the orbits of this $S_4$-set. You do not need Burnside’s formula for this.
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Solution: There are five orbits:

- One consists of the 4 ordered pairs \((i, i, i)\).
- There are three in which two of three coordinates are the same and the third is distinct. Each has 12 elements.
- There is one in which the coordinates are distinct. There are 24 such triples.

These account for all 64 elements in the set.

5. (20 points) How many Sylow 3-subgroups can a finite group \(G\) have if its order is

(a) 21
(b) 39
(c) 51
(d) 90

Solution: By the third Sylow theorem, the number of Sylow 3-subgroups must divide \(|G|\) and be congruent to 1 modulo 3. This means for

(a) it is 1 or 7
(b) it is 1 or 13
(c) it is 1
(d) it is 1 or 10.