This is much longer than the exam will be. The actual midterm will be roughly half this length with similar problems.

1. You are painting octahedral blocks and you have \(n\) colors to choose from for each of the 8 faces. Use Burnside’s formula to determine the number of distinguishable blocks that can be made in this way. How big does \(n\) have to be in order to get 100 distinguishable blocks?

2. Let \(X\) be the set of ordered pairs \((i, j)\) where \(i\) and \(j\) are integers ranging from 1 to 4. Let the symmetric group \(S_4\) act on this set by permuting the integers in the usual way. Describe the orbits of this \(S_4\)-set. You do not need Burnside’s formula for this.

3. Find the center \(Z(G)\) and the commutator subgroup \(C(G)\) for the group \(G = C_5 \times S_3\).

4. Describe the center of every simple
   a. abelian group
   b. nonabelian group
   and prove your answer.

5. Describe the commutator subgroup of every simple
   a. abelian group
   b. nonabelian group
   and prove your answer.

6. Let \(H\) be a normal subgroup of \(G\) of index \(m\). Show that \(g^m \in H\) for every \(g \in G\).

7. Prove that the intersection of two normal subgroups of \(G\) is a normal subgroup.

8. Determine the number of subgroups of order 3 in
   a. \(C_{27} \times C_{27}\)
   b. \(C_9 \times C_9 \times C_9\)
   c. \(C_3 \times C_3 \times C_3\)
9. Describe a nonabelian group of order 55. *Hint:* Consider the group of $2 \times 2$ matrices (under multiplication) of integers modulo 11 of the form
\[
\begin{bmatrix}
  a & b \\
  0 & 1
\end{bmatrix}
\] with $a \neq 0$.

10. Let $S = \{1, 2, 3, 4\}$ and let $X = \mathcal{P}(S)$, the power set of $S$, i.e., the set of subsets of $S$, including the empty set. (There are 16 of them.) The symmetric group $S_4$ acts on $X$ via its action on $S$. Describe the orbits under this action. You do not need Burnside’s formula for this.