1. (20 \text{ points}) \ Describe \ a \ nonabelian \ group \ of \ order \ 21. \ \textit{Hint:} \ Consider \ the \ group \ of \ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \ (\text{under multiplication}) \ of \ integers \ modulo \ 7 \ of \ the \ form \ \text{with} \ a \neq 0.

\textbf{Solution:} \ \text{The} \ \text{matrix} \ \text{group} \ \text{G} \ \text{is} \ \text{nonabelian} \ \text{since} \ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} aa' & ab' + b \\ 0 & 1 \end{bmatrix} \ \text{while} \ \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a'a & a'b + b' \\ 0 & 1 \end{bmatrix} \ \text{and} \ ab' + b \neq a'b + b' \ \text{in general.} \ \text{It} \ \text{has} \ \text{order} \ 42 \ \text{since} \ \text{there} \ \text{are} \ 6 \ \text{possible} \ \text{values} \ \text{of} \ a \ \text{and} \ 7 \ \text{possible} \ \text{values} \ \text{of} \ b. \ \text{It} \ \text{has} \ \text{a} \ \text{normal} \ \text{subgroup} \ \text{B} \ \text{of} \ \text{order} \ 7 \ \text{consisting} \ \text{of} \ \text{matrices} \ \text{with} \ a = 1. \ \text{G}/B \ \text{is} \ \text{isomorphic} \ \text{to} \ \text{the} \ \text{subgroup} \ \text{of} \ \text{order} \ 6 \ \text{consisting} \ \text{of} \ \text{matrices} \ \text{with} \ b = 0, \ \text{which} \ \text{is} \ \text{abelian} \ \text{and} \ \text{cyclic.} \ \text{G}/B \ \text{has} \ \text{a} \ \text{subgroup} \ \text{of} \ \text{order} \ 3 \ \text{in which} \ a \in \{1, 2, 4\}. \ \text{Hence} \ \text{our} \ \text{group} \ \text{of} \ \text{order} \ 21 \ \text{is} \ \text{the} \ \text{set} \ \text{of} \ \text{matrices} \ \text{as} \ \text{above} \ \text{with} \ a \in \{1, 2, 4\}.

2. (20 \text{ points}) \ Determine \ the \ number \ of \ subgroups \ of \ order \ 2 \ in

a. \ C_8 \times C_8

\textbf{Solution:} \ \text{There} \ \text{are} \ \text{three} \ \text{subgroups} \ \text{of} \ \text{order} \ 2, \ \text{namely} \ \langle 4, 0 \rangle, \ \langle 0, 4 \rangle \ \text{and} \ \langle 4, 4 \rangle.

b. \ C_4 \times C_4 \times C_4

\textbf{Solution:} \ \text{There} \ \text{are} \ \text{seven} \ \text{subgroups} \ \text{of} \ \text{order} \ 2, \ \text{namely} \ \langle 2, 0, 0 \rangle, \ \langle 0, 2, 0 \rangle, \ \langle 0, 0, 2 \rangle, \ \langle 2, 2, 0 \rangle, \ \langle 2, 0, 2 \rangle, \ \langle 0, 2, 2 \rangle \ \text{and} \ \langle 2, 2, 2 \rangle.

c. \ C_2 \times C_2 \times C_2

\textbf{Solution:} \ \text{There} \ \text{are} \ \text{seven} \ \text{subgroups} \ \text{of} \ \text{order} \ 2, \ \text{namely} \ \langle 1, 0, 0 \rangle, \ \langle 0, 1, 0 \rangle, \ \langle 0, 0, 1 \rangle, \ \langle 1, 1, 0 \rangle, \ \langle 1, 0, 1 \rangle, \ \langle 0, 1, 1 \rangle \ \text{and} \ \langle 1, 1, 1 \rangle.

\textbf{Note:} \ \text{The} \ \text{Sylow} \ \text{theorems} \ \text{do} \ \text{not} \ \text{apply} \ \text{here} \ \text{because} \ \text{we} \ \text{are} \ \text{not} \ \text{counting} \ \text{Sylow} \ \text{subgroups.} \ \text{Each} \ \text{of} \ \text{the} \ \text{groups} \ \text{is} \ \text{a} \ \text{2-group} \ \text{to} \ \text{start} \ \text{with} \ \text{and} \ \text{is} \ \text{hence} \ \text{equal} \ \text{to} \ \text{its} \ \text{2-Sylow} \ \text{subgroup.}

3. (10 \text{ points}) \ Prove \ that \ the \ intersection \ of \ two \ normal \ subgroups \ H_1 \ and \ H_2 \ of \ G \ is \ a \ normal \ subgroup.

\textbf{Solution:} \ \text{Suppose} \ h \in H_1 \cap H_2. \ \text{Then} \ \text{for} \ \text{every} \ g \in G, \ ghg^{-1} \ \text{is} \ \text{in} \ H_1 \ \text{since} \ H_1 \ \text{is} \ \text{normal,} \ \text{and} \ \text{it} \ \text{is} \ \text{in} \ H_2 \ \text{since} \ H_2 \ \text{is} \ \text{normal.} \ \text{Hence} \ \text{it} \ \text{lies} \ \text{in} \ \text{the} \ \text{intersection} \ H_1 \cap H_2, \ \text{so} \ H_1 \cap H_2 \ \text{is} \ \text{normal.
4. (10 points) Describe the center of every simple
   a. abelian group
   b. nonabelian group
and prove your answer.

   Solution: The center of any group is a normal subgroup. Therefore if $G$ is simple, its center $Z(G)$ is either $G$ or the trivial subgroup $(e)$. For any abelian group $A$ (including the simple case $A = C_p$ for $p$ a prime), $Z(A) = A$. If $G$ is a nonabelian simple group, its center cannot be all of $G$, so it must be the trivial subgroup.

5. (20 points) You are painting tetrahedral blocks and you have $n$ colors to choose from for each of the 4 faces. Use Burnside’s formula to determine the number of distinguishable blocks that can be made in this way. How big does $n$ have to be in order to get 100 distinguishable blocks?

   Solution: The relevant group here is the rotation group of the tetrahedron, which is isomorphic to the alternating group $A_4$. The set $X$ on which it acts is the set of all possible colorings of the 4 faces of the tetrahedron, so $|X| = n^4$. Burnside’s formula tells us that the number of orbits is

   \[ \frac{1}{|A_4|} \sum_{g \in A_4} |X_g|. \]

   The elements of $A_4$ fall into three conjugacy classes: that of the identity element $e$, that of the eight 3-cycles, and that of the three double transpositions. We need to work out the size of the fixed point set $X_g$ in each of these three cases.

   • The identity element $e$ fixes all colorings, so $|X_e| = |X| = n^4$.
   • A 3-cycle corresponds to a 120-degree rotation $\rho_3$ about a vertex and the center of the opposite face. Hence it fixes one face and cyclically permutes the other three. Thus a coloring is fixed if the three permuted faces all have the same color. Thus two colors are possible and $|X_{\rho_3}| = n^2$.
   • A double transposition corresponds to a 180-degree rotation $\rho_2$ about the centers of two opposite edges. It transposes two pairs of sides, so again two colors are possible and $|X_{\rho_2}| = n^2$.

   Combining these we see that the number of orbits is

   \[ \frac{n^4 + 11n^2}{12}. \]

   For $n = 5$ this number is 75, and for $n = 6$ it is 141, so it exceeds 100 when $n \geq 6$.

   Note: There is a 5 point deduction for failure to identify the group correctly. The symmetric group $S_4$ acts if we allow reflections, but a painted block is visually distinguishable from its mirror image. The number of $S_4$-orbits (computed using Burnside’s formula) is

   \[ \frac{n^4 + 6n^3 + 11n^2 + 6n}{24} = \frac{n(n + 1)(n + 2)(n + 3)}{24} = \binom{n + 3}{4}, \]

   which is also (and not coincidentally) the number of degree 4 monomials in $n$ variables.