Math 236H
Final exam
May 4, 2010

Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

1. (10 points) Let \( f(x) = x^4 + 7x^2 + rx + 1 \in \mathbb{Z}[x] \). For which integers \( r \) is \( f(x) \) irreducible? Prove your answer.

2. (10 points) Prove that in a finite \( G \)-set \( X \), if \( g \) and \( g' \) are conjugate elements in \( G \), then their fixed point sets \( X_g \) and \( X_{g'} \) have the same cardinality.

3. (15 points) You are painting blocks that are shaped like regular tetrahedra. Each block has four triangular faces. You have \( n \) colors to choose from for each of the 4 faces. Use Burnside’s formula to determine the number of distinguishable blocks that can be made in this way. How big does \( n \) have to be in order to get 50 distinguishable blocks?

4. (15 points) Prove that every finite integral domain \( D \) is a field.

5. (15 points) Recall that the third Sylow theorem says that if \( |G| = ps \) where \( p \) is prime and does not divide \( s \), then the number \( k_p \) of subgroups of order \( p \) divides \( s \) and is congruent to 1 modulo \( p \). Use it to prove that every group \( G \) of order 665 = 5 \cdot 7 \cdot 19 is cyclic.

6. (10 points) Find the largest integer \( m \) which divides \( n^{13} - n \) for all integers \( n \).

7. (15 points) Let \( G \) be \( C_{30} \) (the cyclic group of order 30) and let \( a \) be a generator (so \( G = \langle a \rangle \)) and let \( e \in G \) be the identity element.

(a) list all elements of \( G \) of order 30.
(b) list all elements of \( G \) of order 10.
(c) list all elements of \( G \) of order 6.
(d) list all elements of \( G \) of order 5.
(e) list all elements of \( G \) of order 4.
(f) list all other elements of \( G \).

8. (10 points) Let \( G \) be a finite group with a subgroup \( H \) such that \( |G| = 2|H| \).

(a) Prove that if \( a \in G \) is not in \( H \), then \( a^2 \in H \). (Hint: it suffices to show that \( a^2H \neq aH \).)
(b) Prove that if \( a \) is not in \( H \), then the order \( a \) is even. (Hint: Show that \( a^n \) is not in \( H \) for any odd integer \( n \).)

9. (10 points) Find all primes \( p \) such that \( x + 5 \) is a factor of \( f(x) = x^4 + x^2 + 1 \) in \( \mathbb{Z}/p[x] \).
10. (15 POINTS) Prove that the symmetric group $S_4$ is generated by the three transpositions $(12)$, $(13)$ and $(14)$.

11. (15 POINTS) Let $p = 2s + 1$ be an odd prime bigger than 3 (so $s > 1$), and let

$$f(x) = \frac{x(x^s + 3p - 1)(x^s + 1)}{3p}.$$ 

Prove that $f(x)$ is an integer whenever $x$ is.

12. (15 POINTS) List the even permutations of order 2 in $S_4$ and say how many there are in $S_5$ and $S_6$.

13. (10 POINTS) Prove that the intersection of two normal subgroups of $G$ is a normal subgroup.

14. (10 POINTS) Determine the number of elements of order 4 in

   a. $C_8 \times C_8$

   b. $C_4 \times C_4 \times C_4$

15. (10 POINTS) Prove that if $n$ is an odd integer, then $n^2 \equiv 1$ modulo 8 and $n^4 \equiv 1$ modulo 16.