1. (10 POINTS) Let \( f(x) = x^4 + rx^3 + 5x^2 + 1 \in \mathbb{Z}[x] \). For which integers \( r \) is \( f(x) \) irreducible? Prove your answer.

2. (10 POINTS) Prove that in a finite \( G \)-set \( X \), if \( g \) and \( g' \) are conjugate elements in \( G \), then their fixed point sets \( X_g \) and \( X_{g'} \) have the same cardinality.

3. (15 POINTS) You are painting blocks that are shaped like triangular prisms. Each block has two triangular faces and three rectangular ones. You have \( n \) colors to choose from for each of the 5 faces. Use Burnside’s formula to determine the number of distinguishable blocks that can be made in this way. How big does \( n \) have to be in order to get 100 distinguishable blocks? (Hint: The relevant group here is \( S_3 \). A prism has 9 edges, three surrounding each triangular face and three others that are parallel and have rectangular faces on either side of them. Each rotation of the prism permutes the three parallel edges.)

4. (10 POINTS) Find the largest integer \( m \) which divides \( n^{21} - n \) for all integers \( n \).

5. (15 POINTS) Prove that every finite integral domain \( D \) is a field.

6. (10 POINTS) Describe the center subgroup of every simple
   (a) abelian group
   (b) nonabelian group
   and prove your answer.

7. (10 POINTS) Show that in the ring \( \mathbb{Z}/p \), \( (a+b)^p = a^p + b^p \) for all \( a, b \in \mathbb{Z}/p \).

8. (15 POINTS) Recall that the third Sylow theorem says that if \( |G| = ps \) where \( p \) is prime and does not divide \( s \), then the number \( k_p \) of subgroups of order \( p \) divides \( s \) and is congruent to 1 modulo \( p \). Use it to prove that every group \( G \) of order 1001 is cyclic.
9. (15 points) Let $G$ be $C_{30}$ (the cyclic group of order 20) and let $a$ be a generator (so $G = \langle a \rangle$) and let $e \in G$ be the identity element.
   (a) list all elements of $G$ or order 30.
   (b) list all elements of $G$ or order 10.
   (c) list all elements of $G$ or order 6.
   (d) list all elements of $G$ or order 5.
   (e) list all elements of $G$ or order 4.
   (f) list all other elements of $G$.

10. (10 points) Let $G$ be a finite group with a subgroup $H$ such that $|G| = 2|H|$.
   (a) Prove that if $a \in G$ is not in $H$, then $a^2 \in H$. (Hint: it suffices to
       show that $a^2H \neq aH$.)
   (b) Prove that if $a$ is not in $H$, then the order $a$ is even. (Hint: Show that
       $a^n$ is not in $H$ for any odd integer $n$.)

11. (10 points) Let $G$ be a group of order $n$. Show that $x^n = e$ (the identity
    element) for every $x \in G$.

12. (15 points) Determine the number of subgroups of order 3 in
    (a) $C_{27} \times C_{27}$
    (b) $C_9 \times C_9 \times C_9$
    (c) $C_3 \times C_3 \times C_3$

13. (10 points) Let $G$ be a group of order $pq$ where $p$ and $q$ are distinct prime
    numbers. (Such a group need not be abelian.) Prove that every proper
    subgroup of $G$ is cyclic.

14. (10 points) Find all primes $p$ such that $x+2$ is a factor of $f(x) = x^6 + x^3 + 1$
    in $\mathbb{Z}/p[x]$.

15. (10 points) Prove that if $n$ is an odd integer, then $n^2 \equiv 1$ modulo 8 and
    $n^4 \equiv 1$ modulo 16.