Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

1. (15 POINTS) Prove that the symmetric group $S_4$ is generated by the three transpositions $(12)$, $(23)$ and $(34)$.

2. (5 POINTS) Eisenstein’s criterion says that the polynomial

$$a(x) = x^n + a_1x^{n-1} + a_2x^{n-2} \ldots + a_n \in \mathbb{Z}[x]$$

is irreducible (over $\mathbb{Z}$) if

(i) each coefficient $a_i$ is divisible by a prime $p$ and

(ii) $a_n$ is not divisible by $p^2$.

Give an example of a polynomial satisfying (i) but not (ii) that is reducible.

3. (15 POINTS) Let $p$ be an odd prime, and let

$$f(x) = x(x^{(p-1)/2} + 1)(x^{(p-1)/2} + p - 1).$$

Prove that $f(x)$ is an integer whenever $x$ is.

4. (10 POINTS) Find a symmetric group $S_n$ with an element of order greater than $2n$ and identify the element.

5. (10 POINTS) Let $G$ be a group of order $pq$ where $p$ and $q$ are distinct primes. Show that every proper subgroup of $G$ is cyclic.

6. (15 POINTS) List the even permutations of order 2 in $S_4$ and say how many there are in $S_5$ and $S_6$.

7. (10 POINTS) You are making necklaces with 7 beads each, and each bead can be any one of $n$ colors. Use Burnside’s formula to find the number $r(n)$ of distinct necklaces. Find the smallest $n$ such that this number exceeds 1000.

8. (10 POINTS) Describe the commutator subgroup $C(G)$ of every simple

a. abelian group

b. nonabelian group

9. (10 POINTS) Prove that the intersection of two normal subgroups of $G$ is a normal subgroup.

10. (10 POINTS) Determine the number of elements of order 4 in

a. $C_8 \times C_8$

b. $C_4 \times C_4 \times C_4$
11. (10 points) Let $n$ be an integer such that $6n - 1$ and $6n + 1$ are both primes. (Such primes are called twin primes.) Show that every group $G$ of order $36n^2 - 1$ is cyclic.

12. (15 points) Prove that every finite integral domain is a field.

13. (10 points) Describe a nonabelian group of order 55. Hint: Consider the group of $2 \times 2$ matrices (under multiplication) of integers modulo 11 of the form

$$M = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \quad \text{with } a \neq 0.$$ 

14. (10 points) Prove that if $n$ is an odd integer, then $n^2 \equiv 1 \pmod{8}$ and $n^4 \equiv 1 \pmod{16}$.

15. (15 points) For which integers $r$ is $f(x) = x^4 + rx^3 + x + 3$ reducible over the integers? Prove your answer.