On this sheet $F$, $E$, $L$ and $K$ always denote fields. Further $F \subseteq E$ or equivalently $E/F$, etc. means that $E$ is a finite field extension of $F$.

1. Let $p(x) \in F[x]$ be a non-constant polynomial of degree $n$. Show that there exists a field $E$ with $F \subseteq E$ and $[E : F] \leq n!$ in which $p(x)$ has all its roots.

2. Let $F \subseteq E$ and let $\alpha$ and $\beta$ in $E$ be algebraic over $F$ with the same minimal polynomial. Show that $F(\alpha) \cong F(\beta)$.

3. Let $p(x)$ and $g(x)$ be in $F[x]$ and let $E$ be any extension field of $F$. Show that if $p(x)$ and $g(x)$ have a common factor $h(x)$ in $E[x]$ with degree $h(x) \geq 1$, then $p(x)$ and $g(x)$ have a common factor in $F[x]$ of degree $\geq 1$.

4. Let $p$ be a prime. Show that every irreducible polynomial in $\mathbb{Z}_p[x]$ divides $x^{pn} - x$ for some $n$.

5. Let $K/F$ be a Galois extension with $\text{Gal}(K/F) = G$. Let $\alpha \in K$ and let $f(x) = \prod_{\sigma \in G} (x - \sigma(\alpha))$. Show
   (a) $f(x) \in F[x]$
   (b) $f(x)$ is a power of the minimal polynomial of $\alpha$ over $F$
   (c) $f(x)$ is the minimal polynomial of $\alpha$ over $F$ if and only if $K = F(\alpha)$

6. Let $K/F$ be a finite field extension. Show that $K/F$ is a Galois extension if and only if $|\text{Gal}(K/F)| = [K : F]$.

7. Let $K/F$ be a Galois extension with $\text{Gal}(K/F) = G$ and assume $G$ is abelian. Show that for any intermediate field $L$, $F \subseteq L \subseteq K$, $L/F$ is Galois.

8. Let $K/F$ be a Galois extension with $\text{Gal}(K/F) = G$. Let $f(x) \in F[x]$ have a root $\alpha \in K$. Prove that $\sigma(\alpha)$ is also a root of $f(x)$ for any $\sigma \in G$.

9. Let $p(x) \in F[x]$ be a separable irreducible polynomial and let $K$ be the splitting field of $p(x)$ over $F$. Assume $\text{Gal}(K/F)$ is an abelian group. Prove that $K = F(\theta)$ where $\theta$ is any root of $p(x)$ in $K$.

10. Let $F_1$ and $F_2$ be subfields of a field $K$ such that $[K : F_1 \cap F_2] < \infty$. If $K/F_1$ and $K/F_2$ are Galois extensions, show that $K/(F_1 \cap F_2)$ is a Galois extension.

11. Let $K/F$ be a Galois extension with $\text{Gal}(K/F) = G$. Let $H_1$ and $H_2$ be subgroups of $G$ and let $L_1$ and $L_2$ be the corresponding intermediate fields. Show that the intermediate field corresponding to $H_1 \cap H_2$ is $F(L_1, L_2)$, i.e. the smallest intermediate field containing both $L_1$ and $L_2$.

12. Show that the only automorphism of $\mathbb{Q}$ is the identity.

13. Let $F \subseteq E$ and let $\alpha, \beta \in E$ be two elements of $E$. Show that if $\alpha + \beta$ and $\alpha \beta$ are both algebraic over $F$, then $\alpha$ and $\beta$ are both algebraic over $F$.

14. Let $K/F$ be a Galois extension with $\text{Gal}(K/F)$ a cyclic group. Show that for any divisor $d$ of $[K : F]$, there exists exactly one intermediate field $L$, $F \subseteq L \subseteq K$ with $[L : F] = d$.

15. Let $F$ be a finite field with $p^n$ elements where $p$ is a prime. Show that if $\alpha \in F$ generates the cyclic group $F^\times$, then the degree of $\alpha$ over $\mathbb{Z}_p$ is $n$.

16. Let $K/F$ be a Galois extension with $\text{Gal}(K/F) = G$. If $\alpha \in K$, prove that $N(\alpha) = \prod_{\sigma \in G} \sigma(\alpha)$ and $\text{Tr}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha)$ are always elements of $F$. 