Let $R$ be a commutative ring with 1. Then an ideal $I$ in $R$ is maximal \( \iff R/I \) is a field.

Prop 7.2.20 Let $K \triangleleft R$ be an ideal. Then

1) Given $I$ be an ideal containing $K$, $I^* = I/K$ is an ideal in $R/K$

2) If $J^*$ is an ideal in $R/K$, then there is an ideal $K \triangleleft J \triangleleft R$
so that $J^* = J/K$

3) If $I$ and $J$ are ideals containing $K$, then $I \subseteq J \implies I/K \subseteq J/K$

Proof of 3: Need to show $I^*$ is closed under addition, subtraction, and multiplication by any element in $R/K$.

This follows from similar calculations in $R$.

2) We have a homomorphism $R \xrightarrow{\phi} R/K$ at $J = \phi^{-1}(J^*) = \{a \in R: \phi(a) \in J^*\}$

It is a subgroup because $\phi$ is a group.
homomorphism, given $a \in J$ and $b \in R$ we need to show $ab \in J$.

$\phi(ab) = \phi(a) \phi(b)$ and $\phi(a) \notin J*$

$\phi(ab) = R/K$

Since $J*$ is an ideal, $\phi(a) \phi(b) \in J*$

Hence $\phi(ab) \in J*$ so $ab \in J$.

(ii) $\Rightarrow$ is clear. For $\exists$

Let $a \in I$ then $a + K \in J/K 

This means

$a + K = b + K$ for some $b \in J$

i.e. $a = b + k$ for $k \in K \in J$
so $a \in J$. This means $I \subseteq J$.

QED

2.7.2 Let $R$ be a commutative ring with 1. Then an ideal $IR$ is maximal if $R/I$ is a field.

Fact: A ring is a field if
it has only two ideals, \((0) = 0, R\).

Suppose \(R/I\) is a field, and let \(J \subset R\) for an ideal \(J\).

\(J/I\) is an ideal in \(R/I\) = field.
so either \(J/I = (0)\) \(\Rightarrow J = I\)
or \(J/I = (1)\) \(\Rightarrow J = R\)

\(R/I\)

Hence \(I\) is maximal by def.

Conversely, suppose \(I\) is maximal.
Let \(J\) be any ideal in \(R/I\).
Then there is an ideal $J$ with $I \subseteq J \subseteq R$ that maps to $J^\times$. This means $J = I$ so $J^\times = J/I = (0)$ on $J = R$ so $J^\times = R/I$.

Hence our ideal $J^\times \subseteq R/I$ is either $(0)$ on $R/I$, so $R/I$ is a field. QED

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Polynomial rings

$R = \text{your favorite ring}$

$R[x,J] = \text{the ring of polynomials}$
Let $R$ be a commutative ring with unity.

Claim: If $I$ is an ideal in $R$ contained in $J$, then $I$ is also contained in $J$.

Proof: Let $a \in I$. Since $I \subseteq J$, we have $a \in J$. Thus, $I \subseteq J$.

Therefore, $I$ can be added with $J$ and the sum is still contained in $J$.
c) If \( R \in \mathbb{N} \), so does \( R[x] \).

1) If \( R \) is a domain (no zero divisors)

then so is \( R[x] \).

Proof of 1):

Let \( 0 \neq f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots \)

with \( a_m \neq 0 \)

and \( 0 \neq g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots \)

with \( b_n \neq 0 \).

Then \( f(x) g(x) = a_m b_n x^{m+n} + \ldots \).

and \( a_m b_n \neq 0 \) so \( f(x) g(x) \neq 0 \).

QED.
Note we can also define
\[ R[x,y] = \text{ring of polynomials in } x \text{ and } y \]
\[ = R[x] [y] \]
\[ = \text{polynomials in } y \text{ whose coefficients are in } R[x]. \]

Def: A \underline{unit} in a ring \( R \) is a \underline{nonzero} element with a multiplicative inverse in \( R \).
The units form a group $\mathbb{R}^*$ under multiplication.

$\mathbb{Z}^* / \{\pm 1\} \cong C_2$

**Proof:** Let $D$ be a domain. Then $D[\mathbb{Z}]^* = D^*$, i.e., the only invertible polynomials over $D$ are constants.

E.g., $2[\mathbb{Z}]^* = \{\pm 1\}$

**Proof:** If $f(x)$ has an inverse $f'(x)$, then $g(x)$ is
also a polynomial.

Each polynomial has a degree, i.e., the largest exponent and \( \deg(f \cdot g) = \deg(f) + \deg(g) \)

and \( \deg(f) \), \( \deg(g) \geq 0 \)

Then \( f \) and \( g \) must have degree 0, i.e., they are constants.

QED

Def. \( \mathbb{R}[x] \) = ring of power series in \( x \) with coeffs in \( \mathbb{R} \)
$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

with $a_i \in \mathbb{R}$

Example \( \ln 2 \in [x] \)

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots
\]

\[
(1-x)(1+x+x^2+\cdots) = 1
\]

\[
2 \cdot [x] \subset \mathbb{Z} \cup \mathbb{Z}[x]/\mathbb{Z}
\]
Division algorithm.
Recall (from grade school) that
\[ a, b > 0 \text{ positive integers} \]
there integers \( q \) and \( r \) with
\[ 0 \leq r < b \]
such that

**Theorem 6.2.2** Division algorithm

\( F \) is a field. Let \( f(x), g(x) \in F[x] \)
and \( g(x) \neq 0 \). Then \( \exists ! q(x) \)
and \( r(x) \) with \( \deg r(x) < \deg g(x) \)
with \( f(x) = q(x)g(x) + r(x) \).
If \( \deg f(x) < \deg g(x) \) then
\[ f(x) = 0 \quad \text{and} \quad r(x) = g(x) \]