1. Consider the polynomial $f(x) = x^4 + 5x^2 + 5$. Let $E$ be the splitting field of $f(x)$ over $\mathbb{Q}$ and let $G = \text{Gal}(E/\mathbb{Q})$.

(a) (10 points) Determine the Galois group $G$.

**Solution:** Let $f(x) = x^4 + ax^2 + b$, so $a = b = 5$. According to the handout on even quartics, the Galois group is $C_4$ because $\sqrt{b} \in \mathbb{Q}(\sqrt{a^2 - 4b})$.

(b) (10 points) Determine the action of $G$ on the zeroes of $f(x)$.

**Solution:** From $f(x) = 0$ we have

$$x^2 = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 5}}{2} = \frac{-5 \pm \sqrt{5}}{2},$$

so the zeros are

$$\pm \sqrt{\frac{-5 \pm \sqrt{5}}{2}}.$$

These are imaginary numbers. Let

$$\alpha = \sqrt{\frac{-5 + \sqrt{5}}{2}} \quad \text{and} \quad \beta = \sqrt{\frac{-5 - \sqrt{5}}{2}}$$

be the square roots with positive imaginary parts. These means their product is a negative real number, namely $-\sqrt{5}$. If $\phi(\alpha) = \beta$, then

$$\phi(\alpha^2) = \beta^2$$

$$\phi\left(\frac{-5 + \sqrt{5}}{2}\right) = \frac{-5 + \phi(\sqrt{5})}{2} = \beta^2 = \frac{-5 - \sqrt{5}}{2}$$

so $\phi(\sqrt{5}) = -\sqrt{5}$ and

$$\phi(\beta) = \phi\left(\frac{-\sqrt{5}}{\alpha}\right) = \frac{\sqrt{5}}{\beta} = -\alpha.$$
(c) (5 points) Find a primitive element of $E$.

**Solution:** Since the order of $G$ is the degree of $f$, any zero of $f(x)$ generates the field $E$.

(d) (5 points) Describe each intermediate subfield.

**Solution:** Since $G = C_4$, there is only one intermediate subgroup and hence only one intermediate field. The subgroup is generated by $\phi^2$ and isomorphic to $C_2$. From the computation above, we see that $\phi^2$ fixes $\sqrt{5}$, so the intermediate subfield is $Q(\sqrt{5})$.

2. (10 points) In his 1300 page mathematical anthology *God Created the Integers*, published in 2007, Stephen Hawking says

To be brief, Galois demonstrated that the general polynomial equation of degree $n$ could be solved by radicals if and only if every subgroup $N$ of the group of permutations $S_n$ is a normal subgroup. Then he demonstrated that every subgroup of $S_n$ is normal for all $n \leq 4$ but not for $n \geq 5$. This demonstrated the impossibility of finding a solution in radicals for all general polynomial equations of degree greater than 4.

Explain why this statement is incorrect.

**Solution:** Hawking has misstated Galois’ main theorem, which says that a polynomial equation of degree $n$ is solvable by radicals if and only if its Galois group, which is a subgroup of $S_n$, is solvable. It need not be a normal subgroup of $S_n$, and it is not true that every subgroup of $S_3$ and $S_4$ is normal.

3. Let $\zeta = e^{2\pi i/11} = \cos(2\pi/11) + i \sin(2\pi/11)$. $E = Q(\zeta)$ is the splitting field for

$$f(x) = (x^{11} - 1)/(x - 1) = \sum_{i=0}^{i=10} x^i.$$ 

Its Galois group $G$ is isomorphic to $C_{10}$, so there are intermediate subfields $K_1$ of degree 2 and $K_2$ of degree 5.

(a) (10 points) Find a generator $\phi$ of $F$ by defining $\phi(\zeta)$. Prove it generates the group by showing it has order 10.

**Solution:** Let $\phi(\zeta) = \zeta^2$. Then $\phi^2(\zeta) = \zeta^4$ and $\phi^5(\zeta) = \zeta^{32} = \zeta^{-1}$. Neither $\phi^2$ nor $\phi^5$ is the identity, so $\phi$ has order 10. The same is true if we define $\phi(\zeta) = \zeta^k$ for $k = 6, 7$ or 8.
(b) (10 points) Find the minimal polynomial \( g(x) \) for the element 
\[ \alpha = 2 \cos(2\pi/11) = \zeta + \zeta^{-1} \]
and explain why it is or is not solvable by radicals. Note that \( \alpha \) is fixed by the subgroup of \( G \) of order 2.

Solution: We have
\[
\begin{align*}
\alpha &= \zeta + \zeta^{-1} \\
\alpha^2 &= \zeta^2 + 2 + \zeta^{-2} \\
\alpha^3 &= \zeta^3 + 3\zeta + 3\zeta^{-1} + \zeta^{-3} \\
\alpha^4 &= \zeta^4 + 4\zeta^2 + 6 + 4\zeta^{-2} + \zeta^{-4} \\
\alpha^5 &= \zeta^5 + 5\zeta^3 + 10\zeta + 10\zeta^{-1} + 5\zeta^{-3} + \zeta^{-5} \\
\alpha^5 + \alpha^4 &- 4\alpha^3 - 3\alpha^2 &- 3\alpha &= \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + \ldots \\
\alpha^5 + \alpha^4 - 4\alpha^3 - 3\alpha^2 + 3\alpha &= \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + \ldots \\
&= -1,
\end{align*}
\]
so
\[ g(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1. \]
It follows that \( \mathbb{Q}(\alpha) = K_2 \) and the Galois group of \( g(x) \) is \( C_5 \), so the equation \( g(x) = 0 \) is solvable by radicals.

(c) (5 points) Do the same for
\[ \beta = \sum_{j=0}^{4} \phi^{2j}(\zeta) = \zeta + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^9. \]
Note that \( \beta \) is fixed by the subgroup of \( G \) of order 5.

Solution: We have
\[
\begin{align*}
\beta^2 &= \zeta^2 + \zeta^6 + \zeta^8 + \zeta^{10} + \zeta^{19} \\
&\quad + 2(\zeta^4 + \zeta^5 + \zeta^6 + \zeta^{10} + \zeta^7 + \zeta^8 + \zeta^{12} + \zeta^9 + \zeta^{13} + \zeta^{14}) \\
&= \zeta^2 + \zeta^6 + \zeta^8 + \zeta^{10} + \zeta^8 \\
&\quad + 2(\zeta^4 + \zeta^5 + \zeta^6 + \zeta^{10} + \zeta^7 + \zeta^8 + \zeta^{1} + \zeta^9 + \zeta^2 + \zeta^3) \\
\beta^2 + \beta &= 3(\zeta + \ldots + z^{10}) = -3,
\end{align*}
\]
so \( g(x) = x^2 + x + 3 \). This equation is solvable since it is quadratic, and \( \beta = (-1 \pm \sqrt{-11})/2 \).

4. (20 points) Let \( f(x) \in F[x] \) be a separable irreducible polynomial with splitting field \( K \) and Galois group \( G \). Assume that \( G \) is abelian. Prove that \( K = F(\theta) \) where \( \theta \) is any zero of \( f(x) \) in \( K \).
Solution: Let $E = K(\theta)$. It is an intermediate field between $F$ and $K$ and is therefore fixed by some subgroup $H$ of $G$. If we can show that $H$ is trivial, then $E = K$ as desired. An element $h \in H$ must fix $\theta$. Let

$$f(x) = (x - \theta_1)(x - \theta_2) \ldots (x - \theta_n) \quad \text{where} \quad \theta_1 = \theta$$

For each $i > 1$, we know there is an element $g_i \in G$ with $g_i(\theta) = \theta_i$. We have

$$g_i h g_i^{-1}(\theta_i) = g_i h (\theta) = g_i (\theta) = \theta_i,$$

so $g_i h g_i^{-1}$ fixes $\theta_i$. Since $G$ is abelian, $g_i h g_i^{-1} = h$, so $h$ fixes each $\theta_i$. This means $h$ is the identity element, so $H$ is the trivial group and $E = K$.

5. Let $f(x) = (x^2 + 1)(x^2 - 2)(x^2 - 3)$, let $E$ be the splitting field of $f(x)$ over $\mathbb{Q}$, and let $G$ be its Galois group.

(a) (5 points) Find $G$ and describe $E$ explicitly.

Solution: The zeros of $f(x)$ are $\pm i, \pm \sqrt{2}$ and $\pm \sqrt{3}$. It follows that $G = C_2^3$ and $E = \mathbb{Q}(i, \sqrt{2}, \sqrt{3})$.

(b) (10 points) List all the quadratic subfields of $E$.

Solution: $G$ has seven subgroups of order 2 and seven subgroups of order 4. The latter each fix a quadratic subfield. The seven quadratic subfields are $\mathbb{Q}(i), \mathbb{Q}(\pm \sqrt{2}), \mathbb{Q}(\pm \sqrt{3})$ and $\mathbb{Q}(\sqrt{\pm 6})$. 