1. (20 points) Let $K/F$ be a Galois extension with $\text{Gal}(K/F) = G$. Let $H_1$ and $H_2$ be subgroups of $G$ and let $L_1$ and $L_2$ be the corresponding intermediate fields. Show that the intermediate field corresponding to $H_1 \cap H_2$ is $F(L_1, L_2)$, i.e. the smallest intermediate field containing both $L_1$ and $L_2$.

Solution: We have $L_1 = K^{H_1}$ and $L_2 = K^{H_2}$. Since smaller subgroups fix larger subfields, $K^{H_1 \cap H_2}$ contains both $L_1$ and $L_2$. To see that it is the smallest such subfield, note that any field containing both $L_1$ and $L_2$ corresponds to a subgroup of $G$ that is contained in both $H_1$ and $H_2$. The largest such group is the intersection $H_1 \cap H_2$, so $K^{H_1 \cap H_2}$ is the smallest such field.

2. (20 points) Let $K/F$ be a Galois extension with $\text{Gal}(K/F) = G$. Let $f(x) \in F[x]$ have a root $\alpha \in K$. Prove that $\sigma(\alpha)$ is also a root of $f(x)$ for any $\sigma \in G$.

Solution: Since $\alpha \in K$ is a root of $f(x)$, $f(x)$ can be factored over $K$ as

$$f(x) = (x - \alpha)g(x) \quad \text{for some } g(x) \in K[x].$$

Applying $\sigma$ fixes $f(x)$ (since it has coefficients in $F$) and gives us another factorization

$$f(x) = (x - \sigma(\alpha))h(x) \quad \text{for some } h(x) \in K[x].$$

This means that $\sigma(\alpha)$ is also a root of $f(x)$.

This is also proved in the book as Corollary 12.1.17.
3. Let \( \zeta = e^{2\pi i/9} = \cos 2\pi/9 + i\sin 2\pi/9 \) and \( \alpha = \zeta + \zeta^{-1} = 2 \cos 2\pi/9 \). Let \( E = \mathbb{Q}(\alpha) \) and \( K = \mathbb{Q}(\zeta) \). We know that \( K \) is a Galois extension of \( \mathbb{Q} \) with \( \text{Gal}(K/\mathbb{Q}) \cong C_6 \).

(a) (5 points) Find a minimal polynomial for \( \alpha \). Note that

\[
\zeta^3 + 1 + \zeta^{-3} = 0.
\]

**Solution:** We have

\[
\alpha &= \zeta + \zeta^{-1} \\
\alpha^2 &= (\zeta + \zeta^{-1})^2 = \zeta^2 + 2 + \zeta^{-2} \\
\alpha^3 &= (\zeta + \zeta^{-1})^3 = \zeta^3 + 3\zeta + 3\zeta^{-1} + \zeta^{-3},
\]

so

\[
\alpha^3 - 3\alpha + 1 = \zeta^3 + 1 + \zeta^{-3} = 0,
\]

so the minimal polynomial is \( f(x) = x^3 - 3x + 1 \).

(b) (10 points) Is \( E \) a Galois extension of \( \mathbb{Q} \)? If it is not, say why, and if it is, find the Galois group.

**Solution:** Since \( \text{Gal}(K/\mathbb{Q}) \) is Abelian, any intermediate field is a Galois extension of \( \mathbb{Q} \). \( E \) is the subfield fixed by the subgroup of \( C_6 \) generated by the automorphism sending \( \zeta \) to \( \zeta^{-1} \). This subgroup has order 2, so \( \text{Gal}(E/\mathbb{Q}) \) is cyclic of order 3.
4. Let \( \zeta = e^{2\pi i/12} = (\sqrt{3} + i)/2 \) and let \( E = \mathbb{Q}(\zeta) \). Note that the minimal polynomial for \( \zeta \) is
\[
f(x) = x^4 - x^2 + 1 = (x - \zeta)(x - \zeta^4)(x - \zeta^7)(x - \zeta^{11}),
\]
and \( E \) is a Galois extension of \( \mathbb{Q} \) with \( \text{Gal}(E/\mathbb{Q}) \cong C_2 \times C_2 \).

(a) (5 points) Find two automorphisms \( \phi_1 \) and \( \phi_2 \) of \( E \) that generate the Galois group and describe their action on the zeroes of \( f(x) \).

Solution: Any two of the three nontrivial automorphisms will do. Each one is determined by what it does to \( \zeta \), so they are described by
\[
\phi_1(\zeta) = \zeta^5 = (-\sqrt{3} + i)/2 \quad \text{with} \quad \phi_1(\zeta^7) = \zeta^{11} \\
\phi_2(\zeta) = \zeta^7 = -\zeta \quad \text{with} \quad \phi_2(\zeta^5) = \zeta^{11} \\
\phi_3(\zeta) = \zeta^{11} = (\sqrt{3} - i)/2 \quad \text{with} \quad \phi_3(\zeta^3) = \zeta^7
\]

(b) (15 points) Describe the subfield of \( E \) fixed by each subgroup of order 2 as \( \mathbb{Q}(\sqrt{d}) \) for some integer \( d \). There are three such subgroups and each is worth five points.

Solution: Let \( H_i \) denote the subgroup generated by \( \phi_i \) for \( i = 1, 2, 3 \).
Let \( \alpha_i = \zeta + \phi_i(\zeta) \) and \( \beta_i = \zeta \phi_i(\zeta) \). Both of these are in \( E^{H_i} \), and if either is not in \( \mathbb{Q} \), it generates the intermediate field. Thus we have
\[
\begin{align*}
\alpha_1 &= \zeta + \zeta^5 = i \quad \text{so} \quad E^{H_1} = \mathbb{Q}(\sqrt{-1}) \\
\alpha_2 &= \zeta + \zeta^7 = 0 \\
\beta_2 &= \zeta^8 = -\zeta^2 \\
&= -3 + 2\sqrt{-3} - 1 \\
&= -1 + \frac{4}{\sqrt{-3}} \quad \text{so} \quad E^{H_2} = \mathbb{Q}(\sqrt{-3}) \\
\alpha_3 &= \zeta + \zeta^{11} = \sqrt{3} \quad \text{so} \quad E^{H_2} = \mathbb{Q}(\sqrt{3})
\end{align*}
\]
5. Let \( f(x) = x^4 - 4x^2 - 5 \) and let \( E \) be the splitting field of \( f(x) \) over \( \mathbb{Q} \).

(a) (5 points) Find the degree \([E : \mathbb{Q}]\).

**Solution:** We can solve the equation \( f(x) = 0 \) as follows.

\[
f(x) = x^4 - 4x^2 - 5 = (x^2 - 5)(x^2 + 1) = (x - \sqrt{5})(x + \sqrt{5})(x - i)(x + i).
\]

This means that \( E = \mathbb{Q}(i, \sqrt{5}) \), which has degree 4.

(b) (5 points) Find the Galois group \( G = \text{Gal}(E/\mathbb{Q}) \).

**Solution:** There are automorphisms \( \phi_1 \) and \( \phi_2 \) defined by

\[
\begin{align*}
\phi_1(i) &= -i \quad \text{and} \quad \phi_1(\sqrt{5}) = \sqrt{5} \\
\phi_2(i) &= i \quad \text{and} \quad \phi_2(\sqrt{5}) = -\sqrt{5}
\end{align*}
\]

Each has order 2 and they commute with each other, so \( G \cong C_2 \times C_2 \).

(c) (10 points) Find a primitive element of \( E \) and its minimal polynomial.

**Solution:** Let \( \alpha = i + \sqrt{5} \). (This is not the only primitive element.) Then

\[
\begin{align*}
\alpha^2 &= -1 + 2i\sqrt{5} + 5 = 4 + 2\sqrt{-5} \\
\alpha^2 - 4 &= 2\sqrt{-5} \\
(\alpha^2 - 4)^2 &= -20 \\
\alpha^4 - 8\alpha^2 + 16 &= -20 \\
\alpha^4 - 8\alpha^2 + 36 &= 0.
\end{align*}
\]

so the minimal polynomial of \( x^4 - 8x^2 + 36 \).

(d) (15 points) Describe the subfield of \( E \) fixed by each proper subgroup of \( G \).

**Solution:** The group \( G \) has three nontrivial elements, \( \phi_1, \phi_2 \) and \( \phi_3 = \phi_1\phi_2 \). As in the previous problem, let \( H_i \) denote the subgroup generated by \( \phi_i \). Then \( \phi_1 \) fixes \( \sqrt{5} \), \( \phi_2 \) fixes \( i \) and \( \phi_3 \) fixes \( \sqrt{-5} \), so

\[
\begin{align*}
E^{H_1} &= \mathbb{Q}(\sqrt{5}) \\
E^{H_2} &= \mathbb{Q}(\sqrt{-1}) \\
E^{H_3} &= \mathbb{Q}(\sqrt{-5}).
\end{align*}
\]