Def: A group $G$ is *solvable* if there exists

$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n = \{e\}$

where each $G_{i+1}$ is normal in $G_i$ and $G_i/G_{i+1}$ is abelian for $0 \leq i \leq n-1$.

Example: $S_n$ and $A_n$ are not solvable for $n \geq 5$.

$S_4$ and $S_2$ are solvable.

Thm 12.5.18 (Kloos' main result): Let

$\text{char } F = 0$, $f(x) \in F[x]$ with
Splitting field $E$. Then $f(x)$ is solvable by radicals (to be explained) \($\iff\)\) $\text{Gal}(E/F)$ is solvable.

``Solvable by radicals'' means there is a tower of fields $F = K_0 \subset K_1 \subset K_2 \cdots \subset K_m = E$ where $K_{i+1}$ is obtained from $K_i$ by adjoining $\sqrt[n]{a}$ for some $a \in K_i$ and some integer $n$. This means there is a formula for the zeros
of $g(x)$ than can be written in terms of radicals, e.g. the quadratic form.

Definition 5.3.6: Given $a, b \in G$, their commutator $[a, b] = a b a^{-1} b^{-1}$. The commutator subgroup $[G, G] = G'$ is the subgroup generated by all commutators.

If $a b = b a$ then $[a, b] = e$. If $G$ is abelian, then $G' = \{e\}$.

Theorem 5.3.7: For $G$ and $G'$ as above
1) $G'$ is normal in $G$

2) $G/G'$ is abelian

3) If $N < G$ is any normal subgroup with $G/N$ abelian, then $G' \subseteq N$.

Proof: 1) \[ [a, b]^{-1} = (ab^{-1}a^{-1})^{-1} \]
\[ = (ab^{-1}a^{-1})^{-1}b^{-1}a^{-1} = b a b^{-1}a^{-1} \]
\[ = [b, a] \]

To show $G'$ is normal
\[ [gag^{-1}, gbq^{-1}] = (gag^{-1}gbq^{-1}) (gag^{-1})^{-1} (gbq^{-1})^{-1} \]

\[ = (gag^{-1}gbq^{-1}) (gag^{-1}gb^{-1}q^{-1}) \]

\[ = ga^{-1}b^{-1}a^{-1}b^{-1}q^{-1} = q^{-1} \text{ for all } a, b \in G^{-1} \]

2) Define words \( xG' \) and \( yG' \)

\[ xG' yG' = x(yG'yC') yG' \]

\[ = xG' C'G' = xyG' \]

\[ yG' xG' = yxG' \text{ similarly} \]
\((xy)(yx)^{-1} \in G'\)

\[ [x, y] \]

\[ x' y' G' = y x' G' \]

\[ G' = (xy)^{-1}(yx)G' \]

\[ = y^{-1}x^{-1}yxG' \]

\[ = [y^{-1}, x^{-1}]G' = G' \]

\[ \Rightarrow x'y'G' = y'x'G' \]

This means \((xG')(yG') = (yG')(xG')\)

so \(G' / G'\) is abelian.
3) If \( G, N \) is abelian, for any \( x, y \in G \),
\[ xN \cdot yN = yN \cdot xN \] so \( [x, y] \in N \)
\[ xyN = yxN \]

Hence \( N \supseteq G' \). \( \Box \) \( \Box \) 

Consider the commutator series
\[ G_1 = G, G_2^{(1)} = G_1^{(1)}, G_2^{(2)} = G_1^{(2)}, \ldots \]
where \( G_2^{(i+1)} \) is the commutator of \( G_2^{(i)} \).

Example: For \( G_1 = A_5 \), \( G_1' = G_1 \). So this process never ends. A gfs being
Perfect if \( G' = G \).

**Theorem 5.3.8** \( G \) is solvable \( \iff G^{(n)} = 1 \) for some \( n \leq \infty \).

**Proof.** Obvious.

\[ G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \ldots \supseteq H_m = 1 \]

where \( H_{i+1} \) is normal in \( H_i \) and \( H_i / H_{i+1} \) is abelian.

We know \( H_1 \supseteq G^{(1)} \). Suppose \( H_k \supseteq G^{(k)} \)

Since \( H_k / H_{k+1} \) is abelian, \( H_k' \subseteq H_{k+1} \).
\[ G^{(k+1)} = (G^{(k)})' \leq H'_k \leq H_{k+1} \]

because \( G^{(k)} \leq H_k \)

Hence we have \( G^{(k)} \leq H_k \) for all \( k \). This means \( G^{(m)} \leq H_m = \Sigma \cdot Z \)

QED

Thm 5, 3, 9: \( S_n \) is not solvable for \( n \geq 5 \).

Let \( H \leq S_n \) be the subgroup containing all 3-cycles. Claim same is true of \( H' \).

Let \( \sigma = (i, j, k) \in H \) \( \overrightarrow{ij} \rightarrow k \)
\[ J = (i, j, k, l, m) \in H \]
\[ \rho = (k, j, m) \in H \]

Then \( G = \{ J, \rho \} \). (CHECK THIS)

Hence \( G \in H \).
Hence \( H \leq S_{(b)}^{m} \) for any \( k \).

and \( S_{m}^{(k)} \neq \{ e \} \) for all \( k \). QED

Same argument works for \( A_{m} \).

Theorem 5.3.112: Let \( G \) be solvable. Then

1) every subgroup of \( G \) is solvable.
2) Let $H \unlhd G$. Can show by induction on $i$ that $H^{(i)} \unlhd G$.

Hence $H^{(n)} > \exists \phi$ for $i \gg 0$ and $H$ is solvable.

2) Let $\phi : G \to M$ be onto

Show $M^{(i)} \subseteq \phi(G^{(i)})$ by induction on $i$.

It is true for $i = 0$.

Given $m, n \in M^{(i)}$, $\exists x, y \in G^{(i)}$

with $\phi(x) = m$ and $\phi(y) = n$.

$M^{(i+1)} \supseteq [m, n] = [\phi(x), \phi(y)] = \phi[x, y] \in \phi(G^{(i)})$.

Ref.
This is the inductive step. 

QED

Thm 5.3.11 If $H \triangleleft G$ is normal, then $G_1$ is solvable $\iff$ $N$ and $G_1/H$ are solvable.

Proof:

$\Rightarrow$ True by 5.3.10

$\Leftarrow$ Suppose $N$ and $G_1/H$ are solvable. Then $H = H_0 \geq H_1 \geq H_2 \geq \ldots \geq H_n = \{e\}$ as usual,

$G_1/H = G_0/H \geq G_1/H \geq \ldots \geq G_n/H = \{e\}$

where $G_i/H$ is normal in $G_i/H$ and
\[ (C_i/14) / (C_{i+1}/14) \text{ is abelian} \]
\[ i \equiv (5n+7) \text{ in } \text{Papa}, \ldots \]
\[ C_i / C_{i+1} \]

So we have

\[ G_i = G_0 \supset G_1 \supset G_2 \ldots \quad G_m = H = H_0 \supset H_1 \supset \ldots \supset H_m = \{e\} \]

\[ G_{i+1} \text{ is normal in } G_i \text{ etc,} \quad \text{so } G \text{ is solvable.} \]

QED

\[ S_4 \text{ is solvable, } S_5 \text{ is not solvable} \]

\[ S_4 \text{ is not normal in } S_5 \text{. Thm 5.3.11} \]

\[ \text{does not apply here.} \]