Def: Let $F \subseteq E$ be a field extension with $G = \text{Gal}(E/F)$. If $E^G$ (subfield of $E$ fixed by $G$) is a proper subfield of $F$, then $E$ is a Galois extension of $F$.

FTGT: Let $F \subseteq E$ be a Galois extension and $G = \text{Gal}(E/F)$. For each intermediate field $F \subseteq K \subseteq E$ let $x(k) = \text{Gal}(E/K)$. 
There is a 1-1 correspondence between each $K$ and subgroup of $G$.

Example 1 (Day 1)

$S = \frac{-1 + \sqrt{3}}{2}$

$F = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{-3})$, $G = S_3$

Splitting field for $x^3 - 2 = \alpha(x)$

$E = \mathbb{Q}(\sqrt[3]{2})$ $\mathbb{Q}(\sqrt[3]{-2})$ $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{-2})$

$G$ permutes the 3 cube roots of 2 \{2^{1/3}, 2^{1/3}, 2^{1/3}\}
Example 2: 
$F = F_p$, $E = F_{p^6}$, $G = C_6$

$p$ is prime

Let $x \in E$ not in $F_{p^2}$ or $F_{p^3}$. $G$ is generated
F \subset K \subset E \quad G = \text{Gal}(E/F) \quad E^G = F

Prop 12.2.2 \quad \text{Gal}(E/K) \supset \text{Gal}(E/F)

Field automorphisms of E that fix K:
\text{Gal}_K(E) \supset \text{Gal}_F(E)

Proof: Follows from definition.

Prop 12.2.3 \quad \text{Assume } K \text{ is the splitting field for } g(x) \in \mathbb{F}[x]. \text{ Then any } f \in G \text{ sends } K \text{ to itself.}
we get an element of $\text{Gal}(K/F)$.

E.g. \( \phi(\sqrt[3]{2}) = 5\sqrt[3]{2} \) and \( \phi(5\sqrt[3]{2}) = 5^2\sqrt[3]{2} \)

Then \( \phi(\frac{5\sqrt[3]{2}}{\sqrt[3]{2}}) = \frac{5^2\sqrt[3]{2}}{\sqrt[3]{2}} \)

\[ \phi(5) = 5 \]

**Proof**

\( K = F(\alpha_1, \alpha_2, \ldots, \alpha_n) \) where the \( \alpha_i \) are the zeros of \( q(x) \). These \( \alpha_i \) are permuted by any \( \phi \in \text{Gal}(E/F) \).

Hence \( \phi(K) = K \). \( \text{QED} \)
Theorem 12.2.24. Assume in addition that $E$ is the splitting field for some $f(x) \in F[x]$. Then

1. $H = \text{Gal}(E/K)$ is a normal subgroup of $G = \text{Gal}(E/F)$.

2. $G/H = \text{Gal}(K/F)$.

Proof:

$G = \text{Gal}(E/F) \xrightarrow{\psi} \text{Gal}(K/F)$

$\psi$ is as in 12.2.3

Its kernel is a normal subgroup of $G$. 
key 4 is the gp of auto of E that fix K, i.e. \( \text{Gal}(E/K) = H \).

so H is normal.

(2) Since \( f(x) \in F[x] \subseteq K[x] \), E is the splitting field of \( f(x) \in K[x] \).

By 10.3.21, each \( \sigma \in H \) there is an automorphism \( \phi \) of \( E \) fixing \( F \) (i.e. \( \phi \in G \)).

Hence \( \text{Gal}(K/F) = G/H \). QED
Prop 12.2, 2. For \( E \) with \( G = \text{Gal}(E/F) \)
for \( H \subseteq G \), let \( E^H = \{ x \in E : \sigma(x) = x \} \) for all \( \sigma \in H \).

Then 1) \( E^H \) is a subfield of \( E \).

2) If \( H_2 \subset H_1 \subset G \), then \( E^{H_1} \subset E^{H_2} \).

Pf 1) \( E^H \) is a subfield because \( H \) is a gp of field automorphisms.

2) Since \( H_1 \) contains \( H_2 \), everything fixed by \( H_1 \) is also
Proved by \[ \phi \circ \psi \circ \phi \|_{C_p} \leq \| \phi \| \| \psi \| \| \phi \| \] 

QED

The fundamental theorem follows.