Def: A group $G$ is solvable if it has a sequence of subgroups
$G_1 = G_0 > G_1 > G_2 > G_3 > \ldots > G_m = 0$
s.t. $G_{i+1}$ is a normal subgroup of $G_i$
and $G_i / G_{i+1}$ is abelian.

Example: $A_5$ is not solvable.
$S_n$ for $n \geq 5$ is not solvable.
Galois Main Theorem (12,5,18) 1832.

Let $F$ be a field of characteristic 0, and $f(x) \in F[x]$ with splitting field $E$.

Then $f(x) = 0$ can be solved by radicals (to be defined) exactly if $\text{Gal}(E/F)$ is solvable.

Fact 1. There are formulas for solving $f(x) = 0$ for $\deg f \leq 4$.

2. There are quintic polynomials (e.g., $x^5 - 10x - 5$) with Galois group $S_5$. 
Things we need to know about solvable groups (Chapter 5)

Definition: For $a, b \in G$, their commutator is $[a, b] = a^{-1}b^{-1}ab$.

If $ab = ba$, then $[a, b] = e$

More of 5.3.6: The commutator subgroup of $G$, $\langle [G, G] \rangle$, is the subgroup generated by all commutators. Call it $G'$.
Theorem 5.3.7  For G and G', as above
1) G' is normal in G
2) G/G' is abelian
3) If N is a normal subgroup of G, and G'/N is abelian, then G' ⊆ N.

Proof: (1) \([a, b]^{-1} = (ab^{-1}a^{-1})^{-1} = (b^{-1}a^{-1}b^{-1}a^{-1})^{-1} = b^{-1}a^{-1}b^{-1}a^{-1} = [b, a]^{-1} = [gag^{-1}, gbg^{-1}]^{-1} = gag^{-1} \cdot gbq^{-1} \cdot (gag^{-1})^{-1}(gbg^{-1})^{-1} = q^{-1}b q^{-1} \cdot (g^{-1}a^{-1}q^{-1})(gb^{-1}g^{-1})
The conjugate of a commutator is another commutator.

This makes $G'$ a normal subgroup.

(2) Let $a, b \in G/G'$ with preimages $a, b \in G$. Then $[a, b] \rightarrow [\alpha, \beta]$. But $[a, b] \in G'$ so $[\alpha, \beta] = e$.

Hence $G/G'$ is abelian.

(3) If $N \leq G$, is normal with $G/N$ abelian. Let $\alpha, \beta \in G/N$ with preimages $a, b \in G$. Then $\alpha \beta \lambda = \beta \alpha \lambda$. Hence $G/N$ is abelian.
\( a, b \in G \), then \([a, b] \to [a, b] \).

Since \( G/N \) is abelian, \([a, b] = e\).

\([a, b] \) is in the kernel of \( G \to G/N \), so \([a, b] \in N \), so \( G' \subset N \).

Q.E.D.

5.3.7 says \( G' \) is the smallest normal subgroup with an abelian quotient.

Then 5.3.8 \( \forall i \), \( G^{(0)} = G \) and \( G^{(i+1)} = \text{commutator subgroup of} \ G^{(i)} \).

\( G = G^{(0)} \supset G^{(1)} \supset G^{(2)} \supset \ldots \).
(The commutation series $G$ is solvable $\iff G^{(n)} = e$ for some finite $n$)

$\Rightarrow$ is obvious.

$\Rightarrow$ Assume $G$ is solvable and $G = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_n = \{e\}$ as in the definition. Hence $H_i \supset G^{(i)}$. Will show by induction on $i$ that $H_i \supset G^{(i)}$. Suppose $G^{(k)} \leq H_k$. Since $H_k / H_{k-1}$ is
abelian \quad H_R \quad (\text{the comm. grp of } H_R)

is contained in \quad H_{k+1} \quad \text{.} \quad \therefore \quad G_{(k+1)} = \left( G_{(k)} \right)' \subseteq H_R \subseteq H_{k+1}

This is the inductive step. \quad \text{QED}.

\textbf{Thm 5.3.9} \quad S_m \text{ is not solvable for } \quad n \geq 5.

\textbf{Proof} \quad \mathcal{H} \triangleleft S_m \text{ be the subgroup generated by all 3-cycles} \quad \mathcal{H} \subseteq \langle (i,j,k) \rangle
\( S = (i, k, l) \)

\( P = (k, j, m) \) with \( i, j, k, l, m \) distinct

Claim that \( H' \) (the comm. subgroup of \( H \)) also contains all 3-cycles:

\( \sigma = [S, P] \) and so on. Each of these is the comm. of the other 2 (on its universe)

This means \( H' = H \) and \( H \) is not solvable
$H \leq S_n \implies H' \leq S_n'$
and
$H = H^{(i)} \leq S_n^{(i)}$

This means $S_n$ is not reducible.

QED.

Remark

$H = A_n$

$A_n$ for $n \geq 5$ is simple, i.e.
its only normal subgroups are itself and $e$. 
Theorem 5.3.10. If $G$ is solvable, so are all of its subgroups and (2) quotients.

Proof: Let $H \leq G_n$. Then $H^{(i)} \leq G^{(i)}$.

So $H^{(i)} = e$ for $i \geq 0$.

H is solvable.