FTGT: Given a Galois extension $E \supseteq F$ with $\text{Gal}(E/F) = G$, there is a 1-1 correspondence between subgroups $H \subseteq G$ and intermediate fields $F \subseteq K \subseteq E$ with

$$K = E^H$$ and $$H = \text{Gal}(E/K).$$

Example 1: Let $s = \sqrt[5]{2}$ and $E = \mathbb{Q}(s) = \text{splitting field for } (x^5 - 1)/(x-1) = x^4 + x^3 + x^2 + x + 1$
$G = C_4$ generated by $4$ with

$4(3) = \begin{cases} 
3^2 & 5 \\
5 & 5^2 \\
5^2 & 3 \\
5^4 & 5
\end{cases}$

Corresponding to $H = C_2 \subset C_4 = G$ we have

a field $K = \mathbb{Q}(\sqrt{-5})$

$H$ is generated by $4^2$. It interchanges

$5$ and $5^4$. Hence $4^2(5+5^4) = 5+5^4$

and $5+5^4 \in K = E^4$
Let \( x = s + s^4 \)

\[
x^2 = (s + s^4)^2 = s^2 + 2s^5 + s^8
\]

\[
= 2^2 + 2 \times 3^3
\]

Recall \( s^5 = 1 \) and \( 1 + s + s^2 + s^3 + s^4 = 0 \)

\[
x^2 + x = s + s^4 + s^2 + s^3 + s^5
\]

\[
= (s + s^2 + s^3 + s^4) + 1
\]

\[
= 1
\]
Since \( x^2 + x - 1 = 0 \), \( k = O(\sqrt{5}) \)

\[
x = \frac{-1 \pm \sqrt{1 + 4}}{2} = -1 \pm \sqrt{5}
\]

\[
x = \frac{5 + 5^4}{2} = -1 \pm \sqrt{5} = 1.618 = \text{Golden Ratio},
\]

\[
\text{Golden Ratio, } x = \frac{5 + 5^{-1}}{2} \quad \Rightarrow \quad 5^2 - x5 + 1 = 0
\]

\[
x^5 = 5^2 + 1 \quad \in K[S]
\]
\[ S = \frac{x + \sqrt{x^2 - 4}}{2} \]

\[ x = \frac{1 + \sqrt{5}}{2} \]

\[ x^2 = 1 + 2\sqrt{5} + 5 = 6 + 2\sqrt{5} \]

\[ x^2 = 4 = \frac{(3 + \sqrt{5})^2}{2} \]
6. Let $\gamma = e^{2\pi i / 17}$

$E = \mathbb{Q}(\gamma)$ is splitting field for $x^{17} - 1 \over x - 1$

$[E : \mathbb{Q}] = 16$

$\text{Gal}(E/\mathbb{Q}) = (\mathbb{Z}/17)^\times = C_{16}$

It is generated by an automorphism $\gamma$ with $\gamma(\gamma) = \gamma^3$
Lattice of subgroups

\[ \mathbb{Z} \leq C_2 \leq C_4 \leq C_8 \leq C_{16} = G \]

What is \( E_{17} \)?

\( H \leq C_8 \) is generated by \( x^2 \) and \( x^2(3) = 3 \)
The orbit of 5 under H is
\[ \frac{2}{3} \rightarrow 5 \rightarrow 8 \rightarrow 13 \rightarrow 21 \rightarrow 32 \rightarrow 51 \rightarrow 80 \rightarrow 121 \rightarrow \ldots \]

Let \( x \) be the sum of these powers of \( \frac{2}{3} \).

Calculation shows
\[ x^2 + 3x - 2 = 0 \]

\[ x = \frac{-3 \pm \sqrt{9+8}}{2} = \frac{-3 \pm \sqrt{17}}{2} \]
\( y = \sqrt{3} + \sqrt[3]{13} + \sqrt[3]{13}^{-1} \in \mathbb{E}_4 \)

Can find a formula for \( y \) itself by solving 4 successive quadratic equations.

3. \( E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, i = \sqrt{-1}) \)

Let \( w = e^{2\pi i/24} = 2^{4\sqrt{3}} \)}
Claim \( E = \mathbb{Q}(\omega) \)

Note \( e^{2\pi i/4} = i \)
\[ e^{2\pi i/8} = \frac{1+i}{\sqrt{2}} \]

Hence \( \mathbb{Q}(e^{2\pi i/8}) = \mathbb{Q}(i, \sqrt{2}) \)
\( \mathbb{Q}(e^{2\pi i/3}) = \mathbb{Q}(\sqrt[3]{3}) \)
So \( E = \mathbb{Q} \left( e^{2\pi i/24} \right) = \mathbb{Q} \left( i, \sqrt{2}, \sqrt{3} \right) = \mathbb{Q} \left( i, \sqrt{2}, \sqrt{3} \right) \)

\[ [E: \mathbb{Q}] = 8 \]

Note \( (x^8 - 1) = (x^4 - 1)(x^4 + 1) \)
\( (x^4 - 1) = (x^2 - 1)(x^2 + 1) \)
\( (x^2 + 1) = (x^4 + 1)(x^8 - x^4 + 1) \)

\( E \) is the splitting field for \( x^8 - x^4 + 1 \)
$G_n = \text{Ker}(E/\mathfrak{P}) = (C_9)^3$

$w \rightarrow w^5$  \hspace{1cm}  $w \rightarrow w^7$  \hspace{1cm}  $w \rightarrow w^{11}$

If $n$ is not divisible by $2$ or $3$, then

$n^2 = 1 \mod 24$, \hspace{1cm} \text{EXERCISE}$