Main Thm 12.5.8

\[ \begin{align*}
F &= \text{field of char } 0 \\
\sigma(x) &\in F[x] \\
E &= \text{splitting field of } F \\
G &= \Gal(E/F)
\end{align*} \]

Then \( f(x) \) is solvable by radicals \( \Rightarrow \) \( G \) is solvable.

Def 12.5.5 \( f(x) \) as above is solvable by radicals if \( E \) is contained in
a radical tower over $F$.

**Lemma** Given $F \subset K$ as above, $F \subset L \subset K$ such that $L$ is a Galois radical extension of $F$. gal

**Proof** $F \underset{\text{GALOIS}}{\supset} E \overset{\text{radical}}{\supset} K \overset{\text{GALOIS}}{\rightarrow} L$

- Splitting of $f(x) \in \mathbb{F}[x]$.
- $\text{Gal}(L/F)$ is solvable.
- $E$ is Gal/F because it is a splitting field.
Lemma 12.5.15

\[ \begin{array}{c}
F \\
\text{Galois extension of } F \quad \text{radical extension of } F \quad H = \text{Gal}(E/F) \quad \text{extension of } F \\
\text{Gal}(L/E) \quad \text{SOLVABLE} \\
\text{Hence } \text{Gal}(L/E) \rightarrow \text{Gal}(E/F) \\
\end{array} \]
3. $L$ which is a solvable Galois extension of $E$ with
   1. $K \subseteq L$
   2. $L$ is Galois over $F$
   3. radical over $E$
   4. $\text{Gal}(L/E)$ is solvable.

Proof: Let $K = E(\sqrt[n]{\beta})$ for $\beta \in E$

$H = \text{Gal}(E/F) = \{ e = g_1, g_2, \ldots, g_m \}$

$g(x) = \Phi_n(x), \prod (x^{n} - \sigma_i(\beta)) \in E[x]$

$\sigma_i(\beta) \in \mathbb{E}$
where $\Theta_n(x) = \text{n th cyclotomic poly} \\
\in \mathbb{Z}[x] \subseteq \mathbb{E}[x]

1) \text{Let } L \text{ be the splitting field of } g(x). \text{ The coeffs of } g(x) \text{ are fixed by } H_j \text{ so } g(x) \in F[x]. \text{ Since } E \text{ is a Galois extension of } F \text{ it is the splitting field for some } h(x) \in F[x].

2) \text{Then } L \text{ is the splitting field of } h(x)g(x), \text{ so it is Galois.}
over $F$.

(i) Let $S = \sqrt{1} = \text{root of } E_n(x)$,

so $E(S)$ is a simple radical extension of $E$.

By 10.3.21, each $\sigma_i \in H$ extends to $\tilde{\sigma}_i^* \in \text{Gal}(L/F)$.

Let $\rho = \sqrt{\theta}$

$$(\tilde{\sigma}_i^*(\rho))^n = \tilde{\sigma}_i^*(\rho^n) = \sigma_{i,}\tilde{b} = \sigma_{i,}\rho \in E$$

so thus $E$ is a zero of $g(x)$. 
The zeros of $g(x)$ have the form $s_i^* \in \mathbb{C}$ for $0 \leq i < n$ and $1 \leq i \leq n$.

Here is the radical tower leading from $E$ to $L$:

$\begin{array}{cccccc}
E_0 & \rightarrow & E_1 & \rightarrow & E_2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \rightarrow & E_1(\mathfrak{p}) & \rightarrow & E_2(\mathfrak{p}_1(\mathfrak{p})) & \rightarrow & \cdots \\
L = E(s_0 \mathfrak{p}, s_1^* \mathfrak{p}, s_2^* \mathfrak{p}, \ldots, s_n^* \mathfrak{p})
\end{array}$
Each extension in this tower is abelian and radical, so $\text{Rad}(L/E)$ is solvable. QED

The $12,5,15$ and $12,5,15$

with $K$ is a radical tower over $E$

(instead of a simple radical extension)

Proof: More of the same.

On $12,5,17$ Go along with $E=F$.

A radical tower $K$ over $E$ is
contained in a Galois radical extension $L$ of $F$, and $Gal(L/F)$ is solvable.

Recall from a week ago.

Given $F \rightarrow K = \text{radical tower} L/E$.

$E$ and $K$ with $L \in$ Galois radical extension of $F$ with solvable Galois $G$. 

$E \rightarrow L \rightarrow K$.
Let $S$ = set of primes that divide $[K:F]$ and $m = \prod_{p \in S} p$

Let $E$ be obtained from $F$ by adjoining $\sqrt[m]{1}$.

In the radical tower of $E$ induced by the one over $F$, each step is Galois with $g \not| C_p$ for some $p$ in $S$. 

Hence we have the main thm. There solution to \( f(x) = 0 \) by radicals (i.e. there is something like the quadratic formula for the roots of \( f(x) \)) \( \Rightarrow \) The splitting field \( K \) of \( f(x) \) is the top of a radical tower over \( F \).

Cor. If the Galois gp of a polynomial \( f(x) \) is not solvable, then...
$f(x)$ cannot be solved by radicals.

Historic Abel (Norwegian, short life) first proved that quintics could not all be solved. 1829?