Let \( w = -\frac{1 + \sqrt{3}}{2} \), so \( w^3 = 1 \).

\[
1 + w + w^2 = 0
\]

\[
w \leftrightarrow w^2 = -1 - w
\]

\[
a + bw \rightarrow a + bw^2 = (a-b) - b \cdot w
\]

\[
a + b \cdot \sqrt{3} \leftrightarrow a - b \cdot \sqrt{3}
\]

\[
\mathcal{O}(\sqrt{3}) = \{ a + b \cdot \sqrt{3} \mid a, b \in \mathbb{Q} \}
\]

\[
L = \mathbb{K}(\sqrt{3}) = \{ c + d \cdot \sqrt{3} + e \cdot \sqrt{4} \mid c, d, e \in \mathbb{K} \}
\]

\( \mathbb{K} \) is a 2-dimensional vector over \( \mathbb{Q} \).
$L$ is $2$-dimensional.

$L$ has a group of field automorphisms isomorphic to $S_3$. This is called the Galois group of $L$ over $\mathbb{Q}$, denoted $\text{Gal}[L/\mathbb{Q}]$.

The 3 cube roots of $2$. 
$w^3, w^2, w^\frac{3}{2}$ are permuted by elements of $G,S_n$.

Consider $2$ diagrams
Subgroups of $G$, $C$

Normal subgroups

Index

Rank $/\mathbb{Q}$

Subfields of $L$
\[ F_1 = \{ (x, y) \in \mathbb{R}^2 : x, y, z \in \mathbb{Q} \} \]
\[ F_2 = \{ (w^3, z) \in \mathbb{Q}^2 : x, y, z \in \mathbb{Q} \} \]
\[ F_3 = \{ (w^2, z) \in \mathbb{Q}^2 : x, y, z \in \mathbb{Q} \} \]
For any subgroup $H < G = S_3$, the set $L^H = \{ \lambda \in \mathbb{L} : h(\lambda) = \lambda \text{ for each } h \in H \}$ is the fixed point field of $H$. If $H$ is a normal subgroup of $G$, then the subfield $L^H$ has an automorphism $g$ isomorphic to $G/H$. 
$K = L^2$ has an aut. of order 2 ($\cong G_2/S_2$) $F_1, F_2, F_3$ have trivial automorphism $g$s.
Other coming attractions

Ruler + compass constructions.

Bisection of an angle

It is possible to construct an \( n \)-sided polygon for any \( n \) of the form
Lema 1.4.\, \textit{Let }$$a \in \mathbb{Z}$$\text{ be a prime number.}

Of course, if \( \ell > 0 \), then \( n = \ell \cdot 2 \). 

\textbf{Proof.} \textbf{Claim.} $$n = \ell \cdot 2$$ \textit{implies }$$n = 1 - \text{even}$$

\begin{align*}
\text{Assume } n & = \ell \cdot 2. \\
\text{Then, for } k = 0, 1, 2, \ldots
\end{align*}
$j$ is a power of 2.

$1 + 2^2$, $1 + 2^4$, $1 + 2^8$ and $1 + 2^{16}$ are primes.

$1 + 2^{32}$ is divisible by 641 (Euler).

These are called Fermat primes.
Other R+C problems

1. Trisect an angle  NO
2. Double the cube  NO
3. Squaring the circle  NO

Galois' rig theorem
There is no general formula for the solution of a degree \( n \) polynomial for \( n \geq 5 \).

Related fact: For \( n \geq 5 \), the alternating \( A_n \) is simple.