Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

1. Give definitions of each of the following. (4 POINTS EACH)
   
   (a) Principal ideal domain
   (b) Splitting field of a polynomial \( f(x) \in F[x] \) for a field \( F \).
   (c) Maximal ideal
   (d) Minimal polynomial over \( F \) of an element \( \alpha \in E \) for fields \( E \) and \( F \) with \( F \subseteq E \)
   (e) Irreducible polynomial

2. Find the minimal polynomial \( f(x) \) for \( \alpha = \sqrt{2} + \sqrt{3} \) over each of the following fields \( F \). (5 POINTS EACH)
   
   (a) \( F = \mathbb{Q} \)
   (b) \( F = \mathbb{Q}(\sqrt{2}) \)
   (c) \( F = \mathbb{Q}(\sqrt{3}) \)
   (d) \( F = \mathbb{Q}(\sqrt{6}) \)

3. (20 POINTS) Let \( R \) be a commutative ring with 1 and let \( I \) be an ideal in \( R \). Show that \( I \) is a prime ideal of \( R \) if and only if \( R/I \) is an integral domain.

4. (20 POINTS) Let \( F \subseteq E \) with \([E : F] = m\). If \( p(x) \in F[x] \) is irreducible over \( F \) of degree \( n \) with \((n, m) = 1\), show that \( p(x) \) has no roots in \( E \).

5. Without using the result of the previous problem, use direct calculation to show (10 POINTS EACH) that
   
   (a) \( x^2 + x + 1 \) has no roots in \( E = \mathbb{Z}/2[y]/(y^3 + y + 1) \), the field with 8 elements.
   (b) \( x^3 - x + 1 \) has no roots in \( E = \mathbb{Z}/3[y]/(y^2 + 1) \), the field with 9 elements.

   HINTS: When working mod \( p \), \( (x + y)^p = x^p + y^p \). Each element \( a \in \mathbb{Z}/p \) satisfies \( a^p = a \).