1. (20 points) Find the minimal polynomial \( f(x) \) for \( \alpha = \sqrt[3]{2} + \sqrt[3]{3} \) over the rationals. You need not prove that your monic polynomial is irreducible.

Solution:

\[
\begin{align*}
\alpha^3 &= \left( \sqrt[3]{2} + \sqrt[3]{3} \right)^3 = 2 + 3\sqrt[3]{12} + 3\sqrt[3]{18} + 3 \\
\alpha^3 - 5 &= 3\sqrt[3]{12} + 3\sqrt[3]{18} = 3\sqrt[3]{6} \left( \sqrt[3]{2} + \sqrt[3]{3} \right) = 3\sqrt[3]{6}\alpha \\
(\alpha^3 - 5)^3 &= \alpha^9 - 15\alpha^6 + 75\alpha^3 - 125 = 162\alpha^3 \\
\alpha^9 - 15\alpha^6 - 87\alpha^3 - 125 &= 0 \\
\text{so } f(x) &= x^9 - 15x^6 - 87x^3 - 125.
\end{align*}
\]

2. (20 points) Let \( F \subseteq E \) with \([E : F] = m\). If \( p(x) \in F[x] \) is irreducible over \( F \) of degree \( n \) with \((n, m) = 1\) and \( n > 1\), show that \( p(x) \) has no roots in \( E \).

Solution: A root of \( p(x) \) must lie in the field \( K = F[x]/(p(x)) \). Since \( p(x) \) has degree \( n \), so \([K : F] = n\). Since \( n \) does not divide \( m \), \( E \) does not have a subfield isomorphic to \( K \), so \( p(x) \) has no roots in \( E \).

3. Give definitions of each of the following. (5 points each)

(a) The Galois group of a field \( E \) over a subfield \( F \).

Solution: See 12.1.14 in the textbook.

(b) The splitting field of a polynomial \( p(x) \in F[x] \) for a field \( F \).

Solution: See 10.3.5 in the textbook.

(c) The minimal polynomial over \( F \) of an element \( \alpha \in E \) for fields \( E \) and \( F \) with \( F \subseteq E \).

Solution: See 10.2.13 in the textbook.

(d) Irreducible polynomial.

Solution: See Theorem 10.2.20 in the textbook.

5. Find a generator of the indicated ideals in the indicated rings. (5 points each)

(a) $\langle x^2 - 1 \rangle \cap \langle x^3 - 1 \rangle$ in $\mathbb{Q}[x]$.

Solution: Since $x^2 - 1 = (x - 1)(x + 1)$ and $x^3 - 1 = (x - 1)(x^2 + x + 1)$, the intersection is

$$\langle (x - 1)(x + 1)(x^2 + x + 1) \rangle = \langle x^4 + x^3 - x - 1 \rangle$$

(b) $\langle x + 1 \rangle \cap \langle x^3 - 1 \rangle$ in $\mathbb{Q}[x]$.

Solution: The gcd of these two polynomials is 1, so the intersection is $\langle (x^3 - 1)(x + 1) \rangle = \langle x^4 + x^3 - x - 1 \rangle$.

(c) $\langle x + 1 \rangle \cap \langle x^3 + 1 \rangle$ in $\mathbb{Z}/(3)[x]$.

Solution: Since $x^3 + 1 \equiv (x + 1)^3$ modulo 3, the intersection is $\langle x^3 + 1 \rangle$.

(d) $\langle x^3 + 1 \rangle \cap \langle x^2 + 1 \rangle$ in $\mathbb{Z}/(2)[x]$.

Solution: Since $x^3 + 1 \equiv (x + 1)(x^2 + x + 1)$ and $x^2 + 1 \equiv (x + 1)^2$ modulo 2, the intersection is

$$\langle (x + 1)^2(x^2 + x + 1) \rangle = \langle (x^2 + 1)(x^2 + x + 1) \rangle = \langle x^4 + x^3 + x + 1 \rangle.$$