1. Give definitions of each of the following. (5 POINTS EACH)

(a) Galois group

Solution: See 12.1.14 in the textbook.

(b) Perfect field

Solution: See 12.1.3 in the textbook.

(c) Galois extension

Solution: See 12.2.10 in the textbook.

(d) Separable polynomial

Solution: See 12.1.3 in the textbook.

2. (20 POINTS) Let $E/F$ be a Galois extension with Galois group $G$, and let $\alpha \in E$. Prove that the elements $N(\alpha) = \prod_{\phi \in G} \phi(\alpha)$ (known as the norm of $\alpha$) and $Tr(\alpha) = \sum_{\phi \in G} \phi(\alpha)$ (known as its trace) are both in $F$.

Solution: Applying any element of $G$ to $N(\alpha)$ will permute its factors and therefore not change its value. Similarly applying it to $Tr(\alpha)$ will permute the summands, and leave the sum fixed. Hence both elements lie in $E^G = F$.

3. (15 POINTS) Let $E$ be a Galois extension of the field $F$ such that $Gal(E/F)$ is abelian. Show that for any intermediate field $K$ with $F \subset K \subset E$, $K$ is a Galois extension of $F$.

Solution: The Fundamental Theorem of Galois Theory says that $K$ is Galois extension of $F$ if and if the subgroup $Gal(E/K)$ of $Gal(E/F)$ is normal. Since $Gal(E/F)$ is abelian, every subgroup of it is normal.
4. Consider the polynomial
\[ f(x) = x^4 - 10x^2 + 22 = (x + \alpha)(x - \alpha)(x + \beta)(x - \beta) \]
where \( \alpha = \sqrt{5 + \sqrt{3}} \) and \( \beta = \sqrt{5 - \sqrt{3}} \), and let \( E = \mathbb{Q}(\alpha, \beta) \) be the splitting field of \( f(x) \) over \( \mathbb{Q} \). The Galois group \( G = Gal(E/\mathbb{Q}) \) is generated by two elements \( \phi_1 \) and \( \phi_2 \) defined by the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \phi_1(x) )</th>
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<tbody>
<tr>
<td>( \alpha )</td>
<td>( -\alpha )</td>
<td>( \beta )</td>
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(a) (5 points) Find the minimal polynomials of \( \beta \) over the field \( \mathbb{Q}(\alpha) \) and of \( \alpha \) over the field \( \mathbb{Q}(\beta) \).

Solution: Note that \( \alpha^2 = 5 + \sqrt{3} \) and \( \beta^2 = 5 - \sqrt{3} \), so \( \alpha^2 + \beta^2 = 10 \). Hence the minimal polynomial of \( \beta \) over the field \( \mathbb{Q}(\alpha) \) is \( x^2 + \alpha^2 - 10 \) and the minimal polynomial of \( \alpha \) over the field \( \mathbb{Q}(\beta) \) is \( x^2 + \beta^2 - 10 \).

(b) (10 points) Describe the Galois group \( G \) as a subgroup of \( S_4 \) by analyzing how it permutes the four roots of \( f(x) \). Determine its order and say whether or not it is abelian.

Hint: Draw a square with vertices labeled \( \alpha, \beta, -\alpha, -\beta \) like this.

Solution: The action of the \( \phi_j \) on the square is via reflections through diagonal and vertical lines. It follows that \( G \) is isomorphic to \( D_8 \), a nonabelian group of order 8.

(c) (10 points) What subgroup fixes the intermediate field \( \mathbb{Q}(\sqrt{66}) \)? Note that \( \sqrt{66} = \alpha \beta (\alpha^2 - \beta^2)/2 \). You should find the image of this element under each \( \phi_j \).

Solution: The following table is helpful.

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<tr>
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<td>( -\sqrt{22} )</td>
<td>( \sqrt{22} )</td>
</tr>
<tr>
<td>( (\alpha^2 - \beta^2)/2 = \sqrt{3} )</td>
<td>( \sqrt{3} )</td>
<td>( -\sqrt{3} )</td>
</tr>
<tr>
<td>( \alpha \beta (\alpha^2 - \beta^2)/2 = \sqrt{66} )</td>
<td>( -\sqrt{66} )</td>
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It follows that the products \( \phi_1 \phi_2 \) and \( \phi_2 \phi_1 \) fix \( \sqrt{66} \). The resulting subgroup is the group of rotations isomorphic to \( C_4 \).
5. Consider the polynomial
\[ f(x) = x^4 - 4x^2 + 1 = (x + \alpha)(x - \alpha)(x + \beta)(x - \beta) \]
where \( \alpha = \sqrt{2 + \sqrt{3}} \) and \( \beta = \sqrt{2 - \sqrt{3}} \), and let \( E = \mathbb{Q}(\alpha, \beta) \) be the splitting field of \( f(x) \) over \( \mathbb{Q} \). The Galois group \( G = Gal(E/\mathbb{Q}) \) is generated by two elements \( \phi_1 \) and \( \phi_2 \) defined by the following table.

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(a) (5 POINTS) Show that \( \mathbb{Q}(\alpha) = \mathbb{Q}(\beta) \), so \( E = \mathbb{Q}(\alpha) \) and \( [E : \mathbb{Q}] = 4 \).

**Solution:**
\[ \alpha \beta = \sqrt{2 + \sqrt{3}} \sqrt{2 - \sqrt{3}} = \sqrt{2^2 - 3} = 1 \]
so \( \beta = \alpha^{-1} \) and they generate the same field.

(b) (5 POINTS) Describe the Galois group \( G \).

**Solution:** The two generators commute and have order 2, so the group is isomorphic to \( C_2 \otimes C_2 \).

(c) (5 POINTS) List the intermediate subfields.

**Solution:** The subgroup generated by \( \phi_1 \) fixes \( \alpha^2 = 2 + \sqrt{3} \), so the corresponding intermediate field is \( \mathbb{Q}(\sqrt{3}) \).
The subgroup generated by \( \phi_2 \) fixes \( \alpha + \beta \), and
\[ (\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 = 4 + 2 = 6, \]
so the corresponding intermediate field is \( \mathbb{Q}(\sqrt{6}) \).
The subgroup generated by \( \phi_1\phi_2 \) fixes \( \alpha - \beta \), and
\[ (\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 = 4 - 2 = 2, \]
so the corresponding intermediate field is \( \mathbb{Q}(\sqrt{2}) \).

(d) (5 POINTS) Show that \( \alpha = (\sqrt{6} + \sqrt{2})/2 \).

**Solution:** Both \( \alpha \) and \( (\sqrt{6} + \sqrt{2})/2 \) are positive real numbers, so it suffices to show that their squares are equal. We know that \( \alpha^2 = 2 + \sqrt{3} \), and
\[ \left( \frac{\sqrt{6} + \sqrt{2}}{2} \right)^2 = \frac{6 + 2\sqrt{12} + 2}{4} = \frac{8 + 4\sqrt{3}}{4} = 2 + \sqrt{3}. \]