1. Give definitions of each of the following. (5 POINTS EACH)
   (a) The Galois group \( \text{Gal}(E/F) \) of a field extension \( F \subset E \)
   (b) A Galois extension.
   (c) A square root tower over a field \( F \).

2. (15 POINTS) Let \( F \subset E \) be a Galois extension with \( G = \text{Gal}(E/F) \). Explain the relation between subgroups \( H \) of \( G \) and intermediate fields \( K \) with \( F \subset K \subset E \).

3. (15 POINTS) Let \( E \) be a Galois extension of the field \( F \) such that \( \text{Gal}(E/F) \) is abelian. Show that for any intermediate field \( K \) with \( F \subset K \subset E \), \( K \) is a Galois extension of \( F \).

4. Consider the polynomial
   \[ f(x) = x^4 + 2x^2 + 3 = (x + \alpha)(x - \alpha)(x + \beta)(x - \beta) \]
   where \( \alpha = \sqrt{-1 + \sqrt{-2}} \) and \( \beta = \sqrt{-1 - \sqrt{-2}} \), and let \( E = \mathbb{Q}(\alpha, \beta) \) be the splitting field of \( f(x) \) over \( \mathbb{Q} \). The Galois group \( G = \text{Gal}(E/\mathbb{Q}) \) is generated by two elements \( \phi_1 \) and \( \phi_2 \) defined by the following table.

   \[
   \begin{array}{c|cc}
   x & \phi_1(x) & \phi_2(x) \\
   \hline
   \alpha & -\alpha & \beta \\
   \beta & \beta & \alpha \\
   \end{array}
   \]

   (a) (5 POINTS) Find the minimal polynomials of \( \beta \) over the field \( \mathbb{Q}(\alpha) \) and of \( \alpha \) over the field \( \mathbb{Q}(\beta) \).

   (b) (10 POINTS) Describe the Galois group \( G \) as a subgroup of \( S_4 \) by analyzing how it permutes the four roots of \( f(x) \). Determine its order and says whether or not it is abelian.
   **HINT:** DRAW A SQUARE WITH VERTICES LABELED \( \alpha, \beta, -\alpha \) AND \( -\beta \) LIKE THIS.

   \[
   \begin{array}{cc}
   \alpha & \beta \\
   -\beta & -\alpha \\
   \end{array}
   \]

   (c) (10 POINTS) What subgroup fixes the intermediate field \( \mathbb{Q}(\sqrt{-6}) \)? Note that \( \sqrt{-6} = \alpha\beta(\alpha^2 - \beta^2)/2 \). You should find the image of this element under each \( \phi_j \).
5. Let \( \zeta = e^{2\pi i / 7} = \cos(2\pi/7) + i \sin(2\pi/7) \) and \( E = \mathbb{Q}(\zeta) \). Note that the minimal polynomial for \( \zeta \) is

\[
f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = (x - \zeta)(x - \zeta^2)(x - \zeta^3)(x - \zeta^4)(x - \zeta^5)(x - \zeta^6)
\]

and \( E \) is a Galois extension of \( \mathbb{Q} \) with \( \text{Gal}(E/\mathbb{Q}) = C_6 \).

(a) (10 POINTS) Find an automorphism \( \phi \) of \( E \) that generates the Galois group and describe its action on the zeros of \( f \).

(b) (10 POINTS) Describe the subfield \( K \) of \( E \) fixed by the subgroup of order 3.

(c) (10 POINTS) Describe the subfield \( L \) of \( E \) fixed by the subgroup of order 2.