Then $H_0 X = \text{free abelian group on the path components of } X$.

**Lemma 1.** If $X$ is path connected then $H_0 X = \mathbb{Z}$.

**Lemma 2.** If $X = A \cup B$ with $A$ and $B$ open in $X$.

Then $H_+ (X) = H_+ (A) \oplus H_+ (B)$.

These 2 imply the theorem.

**Proof of Lemma 2.** Any map $\Delta^n \to X$ has image contained in $A$ or in $B$.

This means $C_+ (X) = C_+ (A) \oplus C_+ (B)$.  


\[ n^* \chi = n^* \alpha \otimes n^* \beta \quad \text{QED} \]

Change of notation: \( S^*_x(X) \) (not \( C^*_x(X) \)) will denote the singular chain complex of \( X \).

Def. Let \( A \subset X \). The relative homology \( H^*_x(X, A) \) is \( H^*(S^*_x(X) / S^*_x(A)) \).

Hence there is SES of chain complexes

\[ 0 \to S^*_x(A) \to S^*_x(X) \to S^*_x(X, A) \to 0 \]

The resulting LES in \( H^*_x \) is the
Exactness Axiom.

The Homotopy Axiom says homotopic maps \( f, g : X \to Y \) induce the same map in \( \text{H}_* \).

Proof: Want to show that \( S_*(f) \) and \( S_*(g) \) are chain homotopic. We have a homotopy:

\[
\begin{array}{c}
\mathbb{E} \\
\downarrow \\
X \\
\downarrow \\
\mathbb{E} \\
\end{array} \quad \Rightarrow \quad 
\begin{array}{c}
Y \\
\downarrow \\
Y \\
\end{array}
\]

where \( \mathbb{E} \) stands for the thickening of \( X \times \{0\} \) and \( \mathbb{E} \times I \).
Let $\Delta^n \to X$ be any map.

$$\text{prinm} = \Delta^n \times I \overset{\sigma \times I}{\longrightarrow} X \times I$$

We will construct a chain $\mathbf{C}$, i.e., a collection

$$\begin{align*}
S_m (X) \xrightarrow{P_m} S_{m+1} (Y) & \quad \text{for } n = 0 \\
\sigma & \quad \Rightarrow P_m (\sigma) \in S_{m+1} (Y)
\end{align*}$$

Need to get from $\sigma$ to a linear $C$

$$\begin{align*}
\Delta^n \times I \overset{\sigma \times I}{\longrightarrow} X \times I \overset{h}{\longrightarrow} Y
\end{align*}$$

Claim that the prism $\Delta^n \times I$ is a union
of (n+1) copies of $\Delta^{n+1}$.

For $n=1$:

Vertices of $\Delta^2$ are

$\{w_0, w_0, w_1, w_1\}$

For $\Delta^1$ they are

$\{w_0, w_1\}$

For $n=2$:

$\Delta^3$ has:

$\{w_0, w_0, w_0, w_1, w_1, w_2, w_2\}$

$\Delta^2$ has:

$\{w_0, w_1, w_2\}$

$\Delta^1$ has:

$\{w_0, w_1\}$
The union of these $3 \Delta^3$'s is $\Delta^2 \times I$.

In general, $\Delta^n_1$ has vertices $\Delta^{n+1}_i$ has vertices $w_0, \ldots, w_i, v_i, \ldots, v_n$ for $0 \leq i \leq n$.

Note that the prism $\Delta^n_x \times I$ has $2(n+1)$ vertices, $\{w_i, v_i : 0 \leq i \leq n\}$.

$\Delta^n_i \xrightarrow{b_i} \Delta^n_x \times I \xrightarrow{\sigma \times I} X \times I \xrightarrow{h} Y$

We define $S_n(X) \xrightarrow{p_n} S_{n+1}(Y)$. 

\[ P_n(0) = \sum_{i=0}^{n} (h^i \circ (0 \times I)) \circ \beta^i \]

Calculation shows that this is a chain homotopy between \( S_x(f) \) and \( S_x(g) \).

QED

**Exercise 1.25** (Thm 2.20) Given spaces

\[ \exists \; A \subset X \text{ with closure } (\overline{A}) \subset \text{interior}(A) \]

Then \( H_x(X - \overline{A}, A - \overline{A}) \to H_x(X, A) \)
(induced by inclusion) is an isomorphism.

Alternate formulation: Given \( A, B \subseteq X \) with \( I = \text{int}(A) \cup \text{int}(B) \), then

\[ H^*_+(B \cup A \cap B) \to H^*_+(X,A) \] is an isomorphism.

Let \( B = X - 2 \) above.

Let \( U = \{ U_1, U_2, \ldots, U \} \) be an open covering of \( X \), i.e., a collection of open subsets whose union is \( X \), i.e., \( \exists \text{int}(A), \text{int}(B) \).
Let $S^U_x(x)$ be the sub-chain complex of $S_x(x)$ generated by "small simplices," i.e. maps $\Delta^n \to U_i$. We will show that $S^U_x(x) \to S_x(x)$ is a chain homotopy equivalence. For $U = \{ \text{int}(A), \text{int}(X \setminus 2) \}$ we get

Thm 2.20.
To replace an arbitrary simplex $(\Delta^n \to X)$ by a sum of small ones, we need a technical tool: BARYCENTRIC SUBDIVISION.

Each vertex is labelled with a subset $\{0,2\}$. $a \in S = \{0,1,2\}$. 

Diagram:

- Labelled vertices: $A, B, C, D, E, F$.
- Subsets: $\{0,1,2\}$, $\{0,2\}$, $\{2\}$.
In general \( n \in \{0, 1, \ldots, n \} \)

vertices of \( A : \{0, 2, 3, 50, 125, 2 \} \)
\( B : \{0, 3, 5, 23 \} \)

Similarly we can subdivide \( \Delta^n \) into \( (n+1)! \) smaller \( \Delta^n \)s.

This process can be iterated.

Union a map \( \Delta^n \rightarrow X \)
If we subdivide enough times, each little $\Delta^n$ will be "small" as above, i.e., it will send it to one of the $U_k$'s. The covering of $I$ pulls back to an open covering of $\Delta^n$ which can be replaced by a finite one since $\Delta^n$ is compact. For some $k$, the $k$th iterated subdivision of $\Delta^n$ makes it a union of
\((n+1)^k\) little \(\Delta^n\), each contained in some \(\sigma^{n-1}(U_\alpha)\).

**Prop 2.21** There are chain homotopy equivalences

\[
\tilde{s}_x^U(X) \iff \tilde{s}_x(X).
\]

**Proof**: Let \(L\) be the inclusion and \(P\) is to be defined with

\[
P|s_x^U = \text{identity}.
\]
$P$ (big simplex) = sum of small simplices obtained by iterated subdivision.

Then

$$S_U^X(x) \xrightarrow{L} S_x^X(x) \xrightarrow{P} S_U^X(x)$$

is the identity

$$S_x^X(x) \xrightarrow{P} S_U^X(x) \xrightarrow{L} S_x^X(x)$$

big $g$ $\rightarrow$ sum of small ones.
This is shown from the equivalence of the identity by tedious calculation. See pp 121-123 of Hatcher. QED

Thm 2.20 follows as a special case.