Def A covering is a map \( \phi : \tilde{X} \to X \) s.t.
each \( x \in X \) has a neighborhood \( U \) s.t.
\( \phi^{-1}(U) \cong U \times D \) where \( D \) is discrete. We
say \( U \) is evenly covered by \( \phi \).

**Examples:**
1. \( X = \mathbb{R} \times D \)  \( \phi = \) projection onto \( \mathbb{R} \)
   \( G = \mathbb{Z} \)
2. \( \mathbb{R} \to S^1 \)  \( \phi(x) = e^{2\pi i x} \) \( G = \mathbb{Z} \) acts by
   translation
3. \( S^n \to \mathbb{R}P^n \)  \( x \to \) line through \( x \) and 0
   \( G = \mathbb{Z}_2 \) acts antipodally
\( \mathbb{I} = \sqrt{3} \sqrt{5} \)

\( \mathbb{G} = \mathbb{C}_5 \)

Diagram:
- \( \mathbb{a} \)
- \( \mathbb{b} \)
- \( \mathbb{c} \)
- \( \mathbb{d} \)
- \( \mathbb{e} \)

\( \mathbb{G} \) acts by rotation.

Diagram:
- \( \mathbb{a}_1 \)
- \( \mathbb{b}_1 \)
- \( \mathbb{a}_2 \)
- \( \mathbb{b}_2 \)
- \( \mathbb{a}_3 \)
- \( \mathbb{b}_3 \)
- \( \mathbb{a}_4 \)
- \( \mathbb{b}_4 \)
- \( \mathbb{a}_5 \)
- \( \mathbb{b}_5 \)
- \( \mathbb{a}_6 \)
- \( \mathbb{b}_6 \)
- \( \mathbb{a}_7 \)
- \( \mathbb{b}_7 \)

Diagram:
- \( \text{Thin 1-30} \)
- \( (Y, y_0) \times \mathbb{I} \) to \( (\bar{X}, \bar{x}_0) \) via covering map
- \( \text{making the diagram commute} \)
- \( \text{incluent} \)
Con $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(X, x_0)$ is 1-1.

**HOMOTOPY LIFTING PROPERTY.**

Def. A **graph** is a 1-dimensional CW-complex, i.e., a space with a discrete set of vertices connected by edges.

**Cycle**

A **tree** is a graph without cycles.

Think of $I$ be a path-connected graph.
with $d_0$ vertices and $d_1$ edges. ($d_1 = d_0 - 1$)

Let $d = d_0 - d_1 \leq 1$. Then

$\pi_1 (X) = F_{1-d} = \text{free gp on } 1-d \text{ generators}$

**Lemma** The above is true for $d_0 = 1$.

**Proof**

$d_0 = 1$ \hspace{1cm} $1 - d = 5$ \hspace{1cm} Use von Kämpen Theorem. QED

$d_1 = 5$

**Proof of theorem** We can construct a tree $T_{CX}$ that contains each vertex of $X$. 
1) Pick a vertex \( v_0 \).
2) Pick an edge meeting \( v_0 \) and call the other vertex \( v_1 \).
3) Pick an edge containing \( v_0 \) and \( v_1 \) and leads to another vertex \( v_2 \).
4) Repeat until we run out of vertices.

A maximal tree above is shown in black. Consider the space \( X/\Gamma \). It is a graph with 1 vertex, so the lemma applies and we
Claim the map \( x \rightarrow x/T \) is a homotopy equivalence. This means \( \pi_1(x) = \pi_1(x/T) = \text{free gp of right rank} \). QED

An: For each \( n > 0 \), \( F_n \) has a rank of \( n \).
Proof: Generalize 4 above. QED

Will prove the following:

Then let $A$ be a path connected (satisfying a hypothesis to be named later) with $\pi_1 X = G$. Then for each subgroup $H \leq G$ there is a covering $\tilde{x}_H \rightarrow \tilde{X}$ with $\pi_1 (\tilde{x}_H) = H$ and $\tilde{x}_H$ path connected. Each path town covering of $\tilde{X}$
is isomorphic to one of these. Moreover, for \( H_1 < H_2 < G \), then we have coverings

\[ X_{H_1} \xrightarrow{b_1} X_{H_2} \xrightarrow{b_2} X_G. \]

If \( H \) is a normal subgroup of \( G \), then \( G/H \) acts on \( \hat{X} \) with orbit \( \hat{X} = \bigcup_{gH} g\hat{X} \).

Conjecture \( \hat{X} = \hat{X}_e \). Then \( G \) acts on \( \hat{X} \).
with $X/G = X$, and for each $H \subset G$,

$X/H = X_H$.

Def $X$ is the universal cover of $X$.

Analogies with Galois theory

$k = \text{field}, \ K = \text{algebraic closure of } k$

$G_\alpha = \text{Gal}(K/k)$. There is a 1-1

correspondance between subfields of $K$

containing $k$ and subgroups of $G$.
with $L = K^H = \text{subfield fixed by } H$.

If $H$ is normal in $G$, then $G/H = \text{Gal}(L, k)$.

Covering spaces $\tilde{X}$. $\tilde{X}_H = \tilde{X}/H$. $G = \pi_1(\tilde{X})$. 

$\tilde{X}_H = \tilde{X}/H$. $G = \pi_1(\tilde{X})$. 

$\tilde{X}$.
Def: A topological space is locally pretty if each nbd $U$ of each $x \in X$ has a pretty subneighborhood.

Def: A space is semi-locally simply connected if each nbd $U$ of each $x \in X$ has a subneighborhood $W$ s.t. $\pi_1(W) \to \pi_1(X)$ is trivial.