Poincaré Duality

Let $M^n$ be a smooth closed (compact and without boundary) $n$-dimensional manifold, i.e. a space in which every point has a nbhd $\cong \mathbb{R}^n$, and given such 2 such nbhd $U$ and $V$ with nonempty intersection.
the maps $W_1 \leftrightarrow W_2$ are $\mathcal{C}^\infty$ or smooth.

Examples: $S^n, S^n \times S^n, CP^n, RP^n$.

A surface is orientable if you can distinguish clockwise + counterclockwise.
$\mathbb{R}^2$ is orientable.

Möbius strip is not orientable.

If $M^n$ is a closed $n$-manifold, consider the $n$-chain $\Delta^n$ which is the sum (up to sign) of all maps $\Delta^n \to M^n$. It may or may not be possible to choose the signs so that $d\eta(x) = 0$. Can also do this with a $\Delta$-complex.
\[ \begin{align*}
\text{1. } d_2(U) &= c \cdot b - a \\
\text{2. } d_2(L) &= a + b - c \\
\text{3. } d_2(U + L) &= 0 \text{ orientable}
\end{align*} \]
\[ d_2(c) = b - a - c \quad d_2(U - L) = 2c - 2L \neq 0 \]

not orientable

(3) not orientable.

Theorem (Poincaré Duality) Let \( M \) be a closed orientable \( n \)-manifold.

Then there are isomorphisms.
\[ H_\cdot(M;\mathbb{Z}) \cong H_{n-4}(M;\mathbb{Z}) \]

Without orientability, we have this for \( A = \mathbb{Z}/2 \), but not for \( A = \mathbb{Z} \).

Example: \( M = \mathbb{RP}^n \). \( M \) is orientable iff \( n \) is odd.

We have a cellular chain \( \ldots \) of the form
\[
\begin{array}{cccccc}
  2 & \hookrightarrow & 2 \\
  0 & \hookrightarrow & 2 & \hookrightarrow & 2 \\
  0 & \hookrightarrow & 2 & \hookrightarrow & 2 & \hookrightarrow & \cdots \\
  & & & & & & 2 \\
  & & & & & & n
\end{array}
\]

\[ H_n(\mathbb{RP}^n) = \begin{cases} 
  \mathbb{Z} & \text{if } n \text{ is odd} \\
  0 & \text{if } n \text{ is even}
\end{cases} \]
When $n=3$,

\[ H_i \begin{pmatrix} 0 & 1 & 2 & 3 \\ \end{pmatrix} \]

and

\[ H, \quad i=1 \rightarrow H^3-i \]

When $n=4$,

\[
\begin{array}{cccccc}
  i & 0 & 1 & 2 & 3 & 4 \\
  H_i & 2 & 2/2 & 0 & 2/2 & 0 \\
  H_{i+1} & 2 & 0 & 2/2 & 0 & 2/2 \\
 \end{array}
\]

It is noted that

\[ H_0 \neq H^4 \\
H_0 \neq H_4 \]

And the coefficients are taken modulo 2:

\[
\begin{array}{cccccc}
  i & 0 & 1 & 2 & 3 & 4 \\
  H_i & 2/2 & 2/2 & 2/2 & 2/2 & 2/2 \\
  H_{i+1} & 2/2 & 2/2 & 2/2 & 2/2 & 2/2 \\
 \end{array}
\]
Poincaré duality and cup products

\[ H^i(M) \otimes H^j(M) \to H^{i+j}(M), \quad i + j \leq n \]

\[ H_{n-i}(M) \otimes H_{n-j}(M) \to H_{n-i-j}(M) \]

The lower map can be interpreted in terms of intersection.

Suppose \( a \in H_{n-i}(M) \) is represented by an oriented submanifold of dimension \( n-i \).
and similarly for $b \in H_{n-i}(M)$ is represented by an oriented $B$. (tangents planes)

Let $x \in A \cap B$. It has no dots in $A, B$ and $M$ homeo to $\mathbb{R}^{n-i} \subset \mathbb{R}^n$.

If they are transverse to each other, the dimension of the intersection is $n-i-j$.

E.g. $n=3$, $i=1$, $j=2$. 
With luck $\partial B$ is an oriented anti-manifold of dim $n-i-j$.

We know have $a \in H_{n-i} M$ and by $A B$,

$b \in H_{n-i} M$

$c \in H_{n-i-j} M$

Poincaré duality gives us classes $\alpha \in H^{i} M$, $\beta \in H^{j} M$ and $\gamma \in H^{i+j} M$ dual to $a$, $b$ and $c$. 
Then \( \delta = \delta \circ \beta \). CUP PRODUCTS ARE POINCARE DUAL TO INTERSECTIONS.

Example: \( M = S^1 \times S^1 \). Will consider some closed curves in it. The hands on a clock define an embedding \( S^1 \xrightarrow{\text{clock}} M \) representing an elt in \( H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \).

\[
\text{CLOCK} = 12a + b
\]
\[ \text{DIAGONAL} = a + b \]
\[ \text{Intersection} = 11 \cdot \text{unit} \in H_0 \]

Let \( a \in H_1 \) and \( b \in H_2 \) be the duals of \( a \) and \( b \). Then:

\[ a^2 = 0, \quad b^2 = 0, \quad ab = -ba = \text{generator of } H_2 \]
\[
M = k \rho \pi
\]

**Are There Any Angular Moments**

For many ambiguous moments:

\[
(x + \delta) (x - \delta) = x^2 - \delta^2 \quad (\forall \delta \geq 0)
\]

Center in chord to 12a + 5
\[ \mathbb{RP}^i \hookrightarrow \mathbb{RP}^n \text{ linear embedding} \]

We also know \[ H^*(\mathbb{RP}^n, \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^{n+1}) \]

for \( x \in H^i \). \( b_i \) is dual to \( x^{n-i} \).

\( b_{n-i} \) is dual to \( x^i \).

\[ \mathbb{RP}^{n-i} \cap \mathbb{RP}^{n-j} \cong \mathbb{RP}^{n-i-j} \]

\[ x^i \cap x^j = x^{i+j} \]

Can do similar example with \( \mathbb{CP}^n \).