Recall a covering is a map \( \tilde{X} \xrightarrow{\tilde{p}} X \) s.t. for each \( x \in X \) there is a nbhd \( U \) s.t. \( \tilde{p}^{-1}(U) \cong U \times D \) for a fixed discrete space \( D \).

We can relax this condition by allowing \( D \) to be any topological space.

**Examples**

1) \( X = D \times \bar{D} \) and \( \tilde{p} = \tilde{p}_2 \) projection to second factor.

2) The Hopf map \( S^3 \xrightarrow{\tilde{p}} S^2 \) with \( \tilde{p}^{-1}(x) \cong S^1 \) for every \( x \in S^2 \). \( \tilde{p}^{-1}(U) \cong S^1 \times U \) for any arbitrary subset \( U \subset S^2 \).
Given two open sets $U_1$ and $U_2$ with $U_1 \cap U_2 \neq \emptyset$, we have:

$$U_1 \times D \xleftarrow{a} a(U_1) \xrightarrow{\pi_1(U_{12})} \pi_1(U_2) \rightarrow \pi_1(U_2) \rightarrow U_2 \times D$$

$$U_1 \xleftarrow{\text{inclusion}} U_1 \cap U_2 \rightarrow U_2$$

$$U_{12} = U_1 \cap U_2 \rightarrow U_2$$

$$U_{12} \times D \xleftarrow{h_1} h_1(U_{12}) \xrightarrow{\pi_1(U_{12})} \pi_1(U_{12}) \rightarrow \pi_1(U_{12}) \rightarrow U_{12} \times D$$

$$(x, d) \rightarrow (x, h_1(d))$$

where $h_1$ is a homeo on $D$.

This data determines the space $X$. 

$X$
We saw in the case of coverings it often happens that \( D = G/H \) where \( G = \prod_{i} X_{i} \) and \( H = \prod_{i} X_{i}^f \) and each \( x_i \) is given by left-null by some element in \( G \). The structure data is a continuous map \( U_1 \cap U_2 \to G \).

**Change of notation**

\[ p'(x) = \begin{array}{ccc}
F & \to & B \\
\downarrow & & \downarrow \\
F' & \to & B' = \text{base space} \\
\end{array} \]

**Fibers** total space

Let \( \phi(t) \) be the homeomorphism of \( F_1 \), so we have structure maps \( U_1 \cap U_2 \to \phi(t) \). It may be the case that \( \phi(t) \) has a subgp \( G \) that always the target of \( G \).
Def: An *n*-dimensional real vector bundle over \( \mathbb{R}^n \) is a fiber bundle with fiber \( \mathbb{R} \) and structure group \( \text{GL}_n(\mathbb{R}) \). i.e., given two overlapping nbds \( U_i \) and \( U_j \), the map \( U_i \cap U_j \to \text{GL}_n(\mathbb{R}) \). Can also define complex vector bundles.

**Example:** \( B = \mathbb{R}P^m \), \( E = \mathbb{R}^m - \{0\} \) and \( \pi : E \to B \) by line through \( -1 \) and \( 0 \). So \( \pi^{-1}(x) = \text{all of } \mathbb{R} \). Want to change this to \( \pi^{-1}(x) = \mathbb{R} - \{0\} \).
Instead let $E = \{ (x, l) \in \mathbb{R}^{n+1} \times \mathbb{RP}^n : x \in L \}$ where $L$ is a line and define \[ p(x, l) = l \] Then $p^{-1}(l) = L \subset \mathbb{R}$.

This is the tautological (or canonical) line bundle $(\mathbb{RP}^n, \mathbb{A})$ has structure \[ O(1) = \{ \xi \in \mathbb{F} \} \] i.e. $E \neq \mathbb{R} \times \mathbb{RP}^n$

where $E' = \{ (x, l) \in \mathbb{R}^{n+1} \times \mathbb{RP}^n : x \in L^\perp \}$

\[ E \subset \mathbb{R}^{n+1} \times \mathbb{RP}^n \subset E' \]

orthogonal complement of $\mathbb{R}^{n+1}$
The 3 spaces $E$, $E'$ and $\mathbb{R}^{n+1} \times \mathbb{RP}^n$ are vector bundles over $\mathbb{RP}^n$ with fibers of dimensions 1, $n$ and $n+1$. Note that over any $x \in \mathbb{RP}^n$, the fiber $\mathbb{R}^{n+1}$ is the direct sum of those in $E$ and $E'$.

Def. An orthogonal $n$-dimensional vector bundle $\overline{E}$ is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $O(n)$.

Two related fiber bundles of interest:

1) There is a $S^{n-1}$-bundle associated with $\overline{E}$. If $\overline{E}$ is the line bundle
over $\mathbb{RP}^n$ above, its unit sphere bundle has total space $S^n$.

2) There is a $\mathbb{B}^n$ bundle associated with $\mathbb{S}$. If $\mathbb{S}$ = line bundle over $\mathbb{RP}^1$, the unit disk bundle has total space a Möbius band.

3) Let $\pi \rightarrow X$ be a space with an $O(n)$-action. Then associated to $\mathbb{S}$ is a fiber bundle $\pi \rightarrow \mathbb{B}$ with fiber $F$.

**Def:** The **Stiefel manifold** $V_{n,k}$ is the set of orthonormal $k$-tuples of vectors in $\mathbb{R}^n$.

It is also the space of $(\mathbb{1}^{k,n})$-matrices.
over $\mathbb{R}$ with orthonormal row vectors. $O(n)$ acts by right multiplication $V_{n,k} = O(n)/O(n-k) = \text{coor space}$

ii) The Grassmann manifold $G_{m,k}$ is the space of $k$-planes through 0 in $\mathbb{R}^n$.

\[ e.g., \mathbb{RP}^n = G_{m,1} \]

Remarks. There is a map $V_{n,k} \xrightarrow{\text{diff.}} G_{m,k}$.

It is a fiber bundle with fiber $O(k)$. $O(k)$ acts on the space $V_{n,k}$ of $(k \times n)$ matrices by left multiplication.
hence \( \mathcal{E}_{m;j,k} = O(n)/O(k) \times O(n-k) \).

Let \( M^n \) be a smooth manifold.

with an embedding \( M^n \hookrightarrow \mathbb{R}^{m+k} \).

Hence we get a map \( M^n \rightarrow \mathbb{R}^{m+k, n} \).

\( X \mapsto \) \( m \)-plane parallel to the tangent space at \( X \).

There is a canonical \( \mathbb{R}^n \)-bundle over \( \mathbb{R}^{m+k, n} \). Its total space is

\[ E = \{(V, X) \in \mathbb{R}^{m+k} \times \mathbb{R}^{m+k, n} \} \]
pullback $= \mathcal{P} \xrightarrow{\pi} E \xrightarrow{\pi} (x, \xi)$

$M^n \xrightarrow{\phi} G_{m+k, n}$

$m \xrightarrow{\pi} \text{tangent plane of } m$

$P = \{ (m, e) \in M \times E : f(m) = f(e) \}$

$P$ is an $\mathbb{R}^n$-bundle over $M$.