Lunch at Cheesecake Factory
Pittsford Plaza noon Sunday
Final Monday at 2:00-5:00 room 3
Previous finals on course website
See notes of 4/30/14 for a review

Both Periodicity Theorems

\[ T_i \quad \text{so } \begin{cases} 2/2 & \text{for } i \equiv 0, 1 \mod 8 \\ 2 & \text{for } i \equiv 3, 7 \\ 0 & \text{else} \end{cases} \quad \text{for odd} \]

\[ T_i : U = \begin{cases} 2 & \text{for odd} \\ 0 & \text{for even} \end{cases} \]

The Hopf-Whitehead J-homomorphism
\[ \pi_k \mathbb{SO}(n) \to \pi_{k+n} S^n \]

An element in \( \pi_k \mathbb{SO}(n) \) is represented by a map \( S^k \to \mathbb{SO}(n) \). We will use it to define a map \( S^k \times D^n \to D^n \to D^n \setminus \partial D^n = S^n \).

\( \mathbb{SO}(n) \) acts on \( D^n \) by rotation. We want a map:

\[ (x,y) \mapsto f(x)(y) \]

The map:

\[ S^k \times D^n \to S^{n+k} \]

\[ R^{k+1} \times R^n - \{0,0^2\} = R^{n+k+1} \]

\[ S^k \times 2D^n \to H \]
Double cone everything

\[ S_{n+k+1} = \sum S_{n+k} \to \sum S_n = S^{n+1} \]

It is known that both vertical maps are isos for \( n > k+1 \) and onto for \( n = k+1 \).
Example \( n = 2, k = 1 \)
\[ \pi_1 \mathbb{S}^0(2) \rightarrow \Gamma_{\infty}(S^2) \]
\[ \Gamma_{\infty}(S^2) = \mathbb{Z} \]

generator \[ \Rightarrow \text{Hopf map} \quad S^3 \rightarrow S^2 \]

Thm (Adams ~ 1965)

a) For \( n > k + 1 \), the image of \( J \) is a direct summand.

b) For \( k = 0, 1 \), \( J \) is \( 1 - 1 \).

c) For \( k = 4n - 1 \), the image is a cyclic \( \mathbb{Z}_p \) of order \( 4n \), where \( 4n \) is the denominator.
of $B_{2n}/2n$, where $B_{2n}$ is the $2n$th Bernoulli number.

$\prod_{k=1}^{n-1} S_{n}$ has a cyclic summand of order $G_{n}$.

What every mathematician should know about Bernoulli numbers:

$x^{p} + y^{p} = 2^{p} \quad p \text{ prime } > 2$

Kummer's Theorem of 1847:

FLT is true for $p$ if $p$ never divides the numerator
of \( B_n/n \). Certain congruences
mean we need only check
roughly \( p \) values of \( n \).
Kummer's condition holds for all
\( p < 100 \) except \( 37, 59 \), and \( 67 \).
Alternate description of
\( A_n = \text{denom of } B_{2^n}/2^n \)

\[ A_n = \gcd \left( k^d \left( k^{2^n} - 1 \right) : k \in \mathbb{Z}, d > 0 \right) \]

\[ n = 1 \quad \gcd \left( 2^d (2^2 - 1), 3^d (3^2 - 1), 4^d (4^2 - 1), \ldots \right) \]

\[ = 24 = A_1 \]
\[ n \rightarrow \Pi_{3+n}(S^n) \rightarrow \frac{1}{2} \cdot \frac{1}{24} \]

\[ n = 2 \rightarrow q_2 = 240 \rightarrow \Pi_{2+n}(S^n) \rightarrow \frac{1}{2} \cdot \frac{1}{240} \]

\[ n = 3 \rightarrow q_3 = 504 \rightarrow \Pi_{3+n}(S^n) \rightarrow \frac{1}{2} \cdot \frac{1}{504} \]

\[ \text{etc.} \]

Other facts about \( \Pi_{n+k}(S^n) \)

Freudenthal Suspension Theorem (1937)

\[ \pi_{k+n}(S^n) \rightarrow \Pi_{n+k+1}(S^n) \text{ is an iso.} \]
for $n > k + 1$

e.g. \[ \pi_3(S^2) \xrightarrow{\pi} \pi_4(S^3) \rightarrow \pi_5(S^4) \]

\[ \mathbb{Z} \xrightarrow{\text{onto}} \mathbb{Z}/2 \xrightarrow{\text{onto}} \mathbb{Z}/2 \]

James Construction

Let $X$ be a space with base pt $x_0$.

Then $J^n X = X^n / \sim$.

If a co-ordinate in $X^n$ is $x_0$, you can transform with the
either adjacent co-ordinates
\((x_1, x_0, x_3) \sim (x_0, x_1, x_3)\n\sim (x_1, x_3, x_0)\)

Exercise \(J^k S^m = (S^m)^k / \sim\)

\[ S^n \cup e_2 n \cup e_3 n \cup \ldots e_k n \]

Then \(\sum J^k X = \sum X \lor \sum (x_1 X) \lor \sum (x_3 X) \ldots \lor \sum (x_k X) \)

James
Recall \( V^n = V \times Y \vee (X \vee Y) \)
\[ \vdash (X \times Y) \vee (X \times \exists x \exists y \exists z \times x \times y) \]
\[ S^m \uplus S^n = S^{m+n} \]

\[ \sum_{j=0}^{k} S^m \leq S^{m+1} \lor S^{2m+1} \lor S^{3m+1} \ldots \lor S^{kn+1} \]

\[ \sum_{j=0}^{\infty} S^m \leq \lor S^{im+1} \]

Then (James \( \sim 1954 \))
\[ \sum_{i \geq 0} X = \sum_{i \geq 0} x \]
e.g. \( \sum_{k=1}^{n} s_k = j \sum_{k=1}^{n} \)

\[ = e^n u e^n u e^n u \ldots \]

What is this good for???

\[ \sum_{k=1}^{n+1} s_k = \sum_{k=1}^{n} s_k \]

\[ = s_{n+1} + s_{n+1} + s_{n+1} \ldots \]

For each degree we have a map

\[ \sum_{k=1}^{n+1} s_k \rightarrow s_{d_{n+1}} \]

JAMES

HOPH

map
It turns out that $S^{n+1} \to S^{2n+1}$ has fiber $S^n$. This leads to long exact sequence

$$\exists \pi_1 S^n \to \pi_1 S^{n+1} \to \pi_1 S^{2n+1} \to \pi_{1+1} S^n \to \pi_{1+1} S^{n+1} \to \pi_{1+1} S^{2n+1} \to \cdots$$

$E$ is the suspension map

$\exists$ is the Heff map
P is for Whitehead product EMP sequence.

This leads to an inductive method of calculating $T^*_k S^n$ starting with our knowledge of $T^*_1 S^1$, i.e.

\[ T^*_k S^1 = \begin{cases} 2 & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases} \]
\[ S \leq S^{n+1} \subseteq \bigcup_{m=1}^{\infty} S^m \]

\[ \sum S^{n+1} \subseteq \sum S^m \]

\[ = S^{n+1} \cup S^{2n+1} \cup S^{3n+1} \cup \ldots \]

We have \[ \sum S^m \to S^{dn+1} \]

In general a map \[ \sum X \to Y \]

is equivalent to a map
\[ X \rightarrow 52Y \]

Collapse green line to pt
red line \( \Rightarrow \) loop in \( Y \)

\[ X \rightarrow 52Y \]

\[ \int_{-m}^{m} S^n \rightarrow S^{d+1} \]
\[ \Omega S^{n+1} \subset J^\infty S^n \longrightarrow S^2 \Sigma^d \]
\[ \text{James-Hopf Map} \]

Why is \( J^\infty S^n \cong \Omega S^{n+1} \)?

Can construct a map
\[ J^\infty S^n \rightarrow S^2 \Sigma^d \]
\[ S^n \rightarrow \Omega S^{n+1} \]
\[ x \longrightarrow \text{longitude} \]
\( S^m \times S^m \rightarrow \prod_{i=1}^{2m} S^n \)

\( (x_1, x_2) \mapsto f(x_1) \times f(x_2) \)

\( (y_1, y_2, \ldots, y_k) \mapsto f(y_1) \times f(y_2) \times \cdots \times f(y_k) \)

\( J\times K \hookrightarrow S \times S^{m+1} \)