Recall a pushout in a category $\mathcal{C}$ is the colimit of a functor $I \to \mathcal{C}$.

$\text{when } I \quad \text{tom} \quad \text{dick} \quad \text{harry}$

The limit is $X$ since $I$ has an initial object.
Let $f : X \rightarrow Y$ be a function. $f$ is a pullback of $g : D \rightarrow C$ along $h : B \rightarrow C$.

Specifically, let $Y = \text{Top}(B)$ and $X = D(h)$.

2.3: $\forall \alpha \in \text{Top}(D(h))$ and $x \in X$, $f^{-1}(\alpha)(x) = \text{pullback of } f \text{ along } h$. 

Diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \text{top} \downarrow \\
D(h) & \xrightarrow{g} & C \\
\end{array}
\]
Then \( \lim (D) = Y_1 \times Y_2 = D(\text{tom}) \times D(\text{mary}) \)

Another application of \( \text{Th. 5} \)

Borsuk-Ulam Theorem. Suppose we have a map \( g: S^2 \rightarrow \mathbb{R}^3 \)

Then \( \exists x \in S^2 \) s.t. \( f(x) = f(-x) \)

Proof: Assume \( \exists \) no such \( x \)
Let \( g(x) = \frac{b(x) - b(-x)}{1 - b(x) - b(-x)} \in S^1 \).

Note \( g(-x) = -g(x) \).

We will show that such a \( g \) does not exist.

Consider \( h : S^1 \to S^2 \) when

\[ \begin{cases} 1 \mapsto S^2 \mapsto S^1 \end{cases} \quad \text{where} \quad h = g \quad \text{and} \quad (\cos 2\pi s, \sin 2\pi s, 0) \leq S^2 < 1 \mathbb{R}^3 \]
This a closed path around the equation:

\[ h(s + \frac{1}{2}) = -h(s) \]

\[ h(1) = h(\frac{1}{2}) + q/2 = h(0) + \frac{q}{2} + \frac{q}{2} \]
\[ \tilde{H}(0) + g = g' \]

Hence our path in \( \mathbb{R} \) goes from 0 to an odd integer.

Hence the closed path \( \tilde{h} \) is essential.

The extension of \( g \) from \( S_1 \) to \( S_2 \) shows that \( \tilde{h} \) is null homotopic.

\textbf{CONTRADICTION.}

Such a \( g \) cannot exist and hence
HAM SANDWICH THEOREM

Given 3 compact subsets of $\mathbb{R}^3$ $K_1, K_2, K_3$ a plane which simultaneously intersects all 3.

Proof Each plane has a unit normal vector. Any plane $P$ is parallel to one that intersects $K_1$. QED
For unit vectors \( x \in S^2 \) we have a plane \( P \) intersecting \( K \). These planes vary continuously with \( x \). Put \( L = \perp \) to \( x \). For \( K_2 \) and \( K_3 \), we can look at the volumes lying above and below \( P \). By taking their differences we get a map \( S^2 \to \mathbb{R}^2 \) \( \beta(-x) = -\beta(x) \). We want an \( x \)
with \( f(x) = (0,0) \). By Remark \( \text{ran} \),

\[ \exists x \text{ with } f(-x) = f(x) \quad \text{so } f(x) = (0,0) \]

as desired. QED.

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**Van Kampen Theorem**

Suppose we have \( X = A \cup B \cup x \cdot A \cap B \) \( \) s.t. \( A \cap B, A, B \) and \( X \) are all path connected. Choose \( x_0 \in A \cap B \)
Then \( \pi_1(X, x_0) = \text{push out of} \)

\( \pi_1(A) \xrightarrow{\alpha_*} \pi_1(A \cap B) \xrightarrow{\beta_*} \pi_1(B) \)

as a group.

**Example**

1. \( \Sigma = \mathbb{RP}^2 \), \( A = \text{Möbius band} M \)
   
   \( B = D^2 \), \( A \cap B = S^1 \)
\[ \pi_1(A) = 2 \quad \pi_1(B) = 0 \quad \pi_1(A \cap B) = \mathbb{Z} \]

The diagram

\[ \pi_1(S^1) = \mathbb{Z} \quad \mathbb{Z} \to \mathbb{Z} = \pi_1(M) \]

Prove that \( n \geq 1/2 \)
2. \( X = S' \cup S' \)

\( A = S' \), \( B = S' \)

\( A \cap B \cong \mathbb{Z} \).

Let \( F_n \) be the free gp on \( n \) generators.
Theorem (to be proved later): \( \forall n \geq 0, F_n \) has a subgraph isomorphic to \( F_{n+1} \).

3. \( X = \text{torus} \)

\[ A \times B \cong S^1 \times I \]

\[ A \cap B \cong D^2 \]

Let \( C = A \cup B \)
\[ D = \mathbb{T}^2 \cup \text{int}(C) \]

\[ C \cap D = D \cap C = S^1 \]

\[ \pi_1(C) = \pi_1(S^1 \times S^1) = \mathbb{F}_2 \]

\[ \pi_1(D) = 0 \]

\[ \pi_1(C \cup B) = \mathbb{Z} \]

\[ x = baba^{-1}a^{-1} \]
The quotient is $G = \mathbb{Z}$, normal subgroup generated by $X$.

In this group

\[ b^{-1}a^{-1}b = c = \text{id} \]
\[ b^{-1}a b = a \]
\[ b^{-1} a = a b \]

So $G = \mathbb{Z} \oplus \mathbb{Z}$.
Lemma: Let \( H \triangleleft G \). The pushout of \( 0 \to H \to G \) is \( G/N \) where \( N \) is the normalizer of \( H \) (the smallest normal subgroup containing \( H \)).

Def: Suppose \( G_0 \to G_1 \) and \( G_0 \to G_2 \) are \( 1-1 \). Then the pushout of \( G_2 \leftarrow G_0 \to G_1 \)
is the the free product with AMALGAMATION $G_1 \times G_2$.

When $G_0 = e$, this is the free product of $G_1$ and $G_2$.

$\mathbb{Z} \times \mathbb{Z} = F_2$. Exercise: Describe $\mathbb{Z}/2 \times \mathbb{Z}/2$.

It is infinite.