1. **Euler characteristic question.** (20 points) Let $X$ be a finite graph with $V$ vertices and $E$ edges. Embed it in $\mathbb{R}^3$ (there is a theorem saying that any graph can be embedded in 3-space; there are some that cannot be embedded in the plane) and let $Y$ be the space of all points within $\epsilon$ (a sufficiently small positive number) of the image of $X$. It is a 3-manifold bounded by a surface $M$. Find the Euler characteristic $\chi(M)$ and prove your answer.

**Hint:** Think of the building set in the lounge, the one with steel balls and black magnetic rods. We are going to build something with $V$ balls and $E$ rods. Find the Euler characteristic of the set of $V$ 2-spheres bounding the $V$ balls. Think about how the Euler characteristic of the surface changes each time you add a rod. *You may use the fact that*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

*under suitable hypotheses on $A$ and $B$.*

**Solution:** The Euler characteristic of the disjoint union of $V$ 2-spheres is $2V$. When we add an edge to the graph, we remove a disk from each of two (not necessarily distinct) spheres. This reduces $\chi$ by two. We then add a cylinder by gluing its two boundary components to the two circles created by removing the two disks. This does not change $\chi$, because both the cylinder and its boundary components have Euler characteristic zero. We do this $E$ times, so $\chi(M) = 2V - 2E$.

2. **Finite graph question.** (20 points) Let $X_1$ be the 1-skeleton of an octahedron, which is a graph with 6 vertices and 12 edges. Let $X_2$ be a graph with 3 vertices and 2 edges connecting each pair of vertices, making 6 edges in all. Let $M_1$ and $M_2$ be the two corresponding surfaces as in the previous problem. Construct maps $X_1 \to X_2$ and $M_1 \to M_2$ which are double coverings.

**Solution:** Embed $X_1$ in $\mathbb{R}^3$ as the edges of the octahedron centered at the origin, with vertices at the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. The group $G = C_2$ acts freely on the complement of the origin (which is homeomorphic to $S^2 \times \mathbb{R}$) by sending $(x, y, z)$ to $(-x, -y, -z)$. The orbit space is $\mathbb{R}P^2 \times \mathbb{R}$. This action preserves the image of $X_1$ and its bounding surface $M_1$. The orbit space $X_1/G$ is a graph with 3 vertices and 6 edges, half the number in $X_1$. Like $X_1$ it has four edges meeting at each vertex, and each edge has distinct endpoints. It follows that it is homeomorphic to $X_2$. It follows that $M_1/G$ is homeomorphic to $M_2$. The desired double coverings are the maps of $X_1$ and $M_1$ to their orbit spaces.
3. Infinite graph question. (30 points) Consider the infinite graph $K$ in $\mathbb{R}^3$ with vertex set

$$\{(i, j, k) \in \mathbb{R}^3 : i, j, k \in \mathbb{Z}\} \cup \left\{\left(\frac{2i+1}{2}, \frac{2j+1}{2}, \frac{2k+1}{2}\right) : i, j, k \in \mathbb{Z}\right\}$$

in which each vertex of the form $(x, y, z)$ is connected by an edge to the eight neighboring vertices

$$\left\{(x \pm \frac{1}{2}, y \pm \frac{1}{2}, z \pm \frac{1}{2})\right\}.$$

Thus the center of each edge is a point in the set

$$\left\{\left(i \pm \frac{1}{4}, j \pm \frac{1}{4}, k \pm \frac{1}{4}\right) : i, j, k \in \mathbb{Z}\right\}.$$

The two endpoints for such an edge with a given combination of signs are

$$(i, j, k) \quad \text{and} \quad \left(i \pm \frac{1}{2}, j \pm \frac{1}{2}, k \pm \frac{1}{2}\right)$$

with the same combination of signs in the second point. There is a corresponding surface $M$ as in the previous two problems.

The group $G = \mathbb{Z}^3$ acts freely $\mathbb{R}^3$ by translation, with $(i, j, k) \in \mathbb{Z}^3$ sending $(x, y, z) \in \mathbb{R}^3$ to $(x+i, y+j, z+k)$. Hence it acts freely on both $K$ and $M$. Describe the finite orbit graph $K/G$ and find the genus of the compact orbit surface $M/G$. Both $K/G$ and $M/G$ are contained in the 3-dimensional torus $\mathbb{R}^3/G \cong S^1 \times S^1 \times S^1$, which is also a quotient of the unit cube.

**Solution:** The orbit graph has two vertices, the orbits of

$$(0,0,0) \quad \text{and} \quad \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

They are connected to each other by 8 edges, the orbits of the ones centered at the points

$$\left(\pm \frac{1}{4}, \pm \frac{1}{4}, \pm \frac{1}{4}\right).$$

hence $V = 2$ and $E = 8$. The result problem 1 implies that $\chi(M) = 2V - 2E = -12$, so the genus of $M$ is 7.

Suppose we take the cube $[-1/2, 1/2]^3$ as a fundamental domain for the group action on $\mathbb{R}^3$. Then the point $(0,0,0)$ is its center and each vertex maps to the orbit of $(1/2, 1/2, 1/2)$. The edges of $K/G$ correspond to the 8 lines connecting the center of the cube to the cube’s vertices.
4. (20 points) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk $D^2$ to itself has a fixed point. You may assume $\pi_1 S^1 = \mathbb{Z}$.

**Solution:** See page 32 of Hatcher.

5. (30 points) Let $M_g$ be a closed oriented surface of genus $g$. Its homology is as follows.

$$H_i(M_g) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}^{2g} & \text{for } i = 1 \\ \mathbb{Z} & \text{for } i = 2 \\ 0 & \text{for } i > 2 \end{cases}$$

Let $M_{g,k}$ be $M_g$ with $k$ disjoint open disks removed. Compute $H_*(M_{g,k})$ for $k > 0$ and prove your answer.

**Solution:**

We use the Mayer-Vietoris sequence in which $A = M(g,k)$, $B$ is $k$ copies of $D^2$ and $C = A \cap B$ is $k$ copies of $S^1$. Then we have

$$
\cdots \longrightarrow H_2(C) \overset{\partial_2}{\longrightarrow} H_2(A) \oplus H_2(B) \overset{\partial_1}{\longrightarrow} H_2(M_g) \overset{\partial_2}{\longrightarrow} \\
\cdots \longrightarrow H_1(C) \overset{\partial_1}{\longrightarrow} H_1(A) \oplus H_1(B) \overset{\partial_1}{\longrightarrow} H_1(M_g) \overset{\partial_1}{\longrightarrow} \\
\cdots \longrightarrow H_0(C) \overset{\partial_1}{\longrightarrow} H_0(A) \oplus H_0(B) \overset{\partial_1}{\longrightarrow} H_0(M_g) \longrightarrow 0
$$

Now $M_{g,k}$ is not a closed manifold, so $H_2(M_{g,k}) = 0$ and $\partial_2$ is one-to-one. It is path connected so $H_0(M_{g,k}) = \mathbb{Z}$ and $\partial_1 = 0$. It follows that

$$H_1(M_{g,k}) = \mathbb{Z}^{2g+k-1}.$$