Math 443 — Final exam

May 6, 2013

Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

1. (20 POINTS) The 5-lemma says that given a commutative diagram of abelian groups with exact rows,

$$
\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\epsilon} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E',
\end{array}
$$

if α, β, δ and ϵ are isomorphisms, then so is γ. Show by counterexample that the triviality of α, β, δ and ϵ does *not* imply the triviality of γ.

> **Solution:** Let p by a prime. One counterexample is
>
> $$
> \begin{array}{ccccccccc}
> 0 & \xrightarrow{i} & \mathbf{Z}/p & \xrightarrow{j} & \mathbf{Z}/p^2 & \xrightarrow{k} & \mathbf{Z}/p & \xrightarrow{\ell} & 0 \\
> \downarrow{0} & & \downarrow{0} & & \downarrow{p} & & \downarrow{0} & & \downarrow{0} \\
> 0 & \xrightarrow{i'} & \mathbf{Z}/p & \xrightarrow{j'} & \mathbf{Z}/p^2 & \xrightarrow{k'} & \mathbf{Z}/p & \xrightarrow{\ell'} & 0.
> \end{array}
> $$

2. (20 POINTS) Let X be a finite CW-complex which is the union of two sub-CW-complexes A and B such that the intersection $A \cap B$ is also a sub-CW-complex. Show that the Euler characteristic $\chi(X)$ satisfies the formula

$$
\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).
$$

> **Solution:** Let x_i, a_i, b_i and c_i denote the number of i-cells in X, A, B and $A \cap B$ respectively. Each cell of X lies in either A or B or possibly both. This means that $x_i = a_i + b_i - c_i$; we subtract c_i so as not to count cells in the intersection twice. It follows that
>
> $$
> \begin{aligned}
> \chi(X) &= \sum_{i \geq 0} (-1)^i x_i \\
> &= \sum_{i \geq 0} (-1)^i (a_i + b_i - c_i) \\
> &= \sum_{i \geq 0} (-1)^i a_i + \sum_{i \geq 0} (-1)^i b_i - \sum_{i \geq 0} (-1)^i c_i \\
> &= \chi(A) + \chi(B) - \chi(A \cap B).
> \end{aligned}
> $$

3. (10 POINTS) Let $M_{g,k}$ be a surface of genus g with k disjoint open disks removed. Assume that the closures of these disks are also disjoint. Use the previous problem to find $\chi(M_{g,k})$.

> **Solution:** We know that $\chi(M_g) = 2 - 2g$, where M_g denotes the surface with genus g. Let $A = M_{g,k}$ and let B be the closure of the k open disks. Then $M_g = A \cup B$ and $A \cap B$ is the disjoint union of k circles. Hence we have
>
> $$\begin{aligned} 2 - 2g = \chi(M_g) &= \chi(A) + \chi(B) - \chi(A \cap B) \\ &= \chi(M_{g,k}) + k - 0 \end{aligned}$$
>
> so $\chi(M_{g,k}) = 2 - 2g - k$.

4. (20 POINTS) Let K be the complete graph on five vertices, meaning that there is a single edge connecting each pair of vertices. Use an Euler characteristic argument to prove that K cannot be embedded in the plane. HINT: SHOW THAT EACH FACE MUST BE BOUNDED BY AT LEAST 3 EDGES.

> **Solution:** A face cannot be bounded by just two edges, because they would have to connect the same pair of vertices. A spherical polyhedron with 5 vertices and 10 edges must have 7 faces in order to have Euler characteristic 2. The hint implies that $E \geq 3F/2$ since each edge belongs to 2 faces. This is a contradiction.

5. (30 POINTS) Let $X_0 \subset \mathbf{R}^3$ be the set of points (x, y, z) in which two of the three coordinates are integers. It is an infinite union of lines parallel to the coordinate axes, and each point with three integer coordinates is the intersection of three such lines.

   Choose a number ϵ with $0 < \epsilon < 1/2$ and let $X_1 \subset \mathbf{R}^3$ be the set of points whose distance from X_0 is $\leq \epsilon$. It is a 3-manifold whose boundary X_2 is a noncompact surface.

   Let $G = \mathbf{Z}^3 \subset \mathbf{R}^3$. It acts freely on all three spaces by translation, and each orbit space is compact.

   (a) Decscribe the graph X_0/G and compute its Euler characteristic and fundamental group.
   (b) Find the genus of the surface X_2/G.

   HINT: Consider the intersection of each X_i with the cube $[-1/2, 1/2]^3$. Then $\mathbf{R}^3/G$ is the 3-torus obtained by making appropriate identifications on this cube. The orbit spaces X_0/G and X_2/G can be studied in a similar way.

> **Solution:**
>
> (a) X_0/G has one vertex, the image of $(0, 0, 0)$, and three edges, the images of the three coordinate axes. Hence its Euler characteristic is -2 and its fundamental group is free on 3 generators.
>
> (b) X_1/G is the 3-manifold obtained by attaching three handles to D^3 and its boundary X_2/G has genus 3.