Math 443 **Final exam**

May 6, 2014

Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

1. **Chain complex question.** (20 POINTS) Suppose we have a long exact sequence of abelian groups of the form

$$0 \longleftarrow C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} \cdots \xleftarrow{d_n} C_n \longleftarrow 0.$$

Let $C(i,j)$ (for $0 \le i \le j \le n$) be the chain complex defined by

$$C(i,j)_k = \begin{cases} C_k & \text{for } i \le k \le j \\ 0 & \text{otherwise} \end{cases}$$

with the same boundary operator as above. Describe $H_*(C(i,j))$.

Solution: We have $H_k(C) = \ker d_k / \operatorname{im} d_{k+1}$. This is 0 for all $i < k < j$ by exactness. At the extreme values of k we have

$$\begin{aligned} H_i C(i,j) &= C_i / \operatorname{im} d_{i+1} = \operatorname{coker} d_{i+1} \\ H_j C(i,j) &= \ker d_j. \end{aligned}$$

2. **Projective plane question.** Let $X = \mathbf{R}P^2$ and let X^k denote the k-fold Cartesian product of X.

(a) (5 POINTS) Find $H_*(X^2; \mathbf{Z}/2)$. Recall that homology with field coefficients converts Cartesian products to tensor products.

Solution:

$$\begin{aligned} H_*(X^2; \mathbf{Z}/2) &= H_*(X; \mathbf{Z}/2) \otimes H_*(X; \mathbf{Z}/2) \\ H_n(X^2; \mathbf{Z}/2) &= \bigoplus_{i+j=n} H_i(X; \mathbf{Z}/2) \otimes H_j(X; \mathbf{Z}/2) \\ &= \begin{cases} \mathbf{Z}/2 & \text{for } n = 0 \\ (\mathbf{Z}/2)^2 & \text{for } n = 1 \\ (\mathbf{Z}/2)^3 & \text{for } n = 2 \\ (\mathbf{Z}/2)^2 & \text{for } n = 3 \\ \mathbf{Z}/2 & \text{for } n = 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) (5 POINTS) Find $H_*(X^2; \mathbf{Z})$.

Solution: Here we have to use the Künneth formula, which says that

$$H_n(X^2;\mathbf{Z}) = \bigoplus_{i+j=n} H_i(X;\mathbf{Z}) \otimes H_j(X;\mathbf{Z}) \oplus \bigoplus_{i+j=n-1} \mathrm{Tor}\,(H_i(X;\mathbf{Z}), H_j(X;\mathbf{Z}))$$

Then we have

$$\bigoplus_{i+j=n} H_i(X;\mathbf{Z}) \otimes H_j(X;\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } n=0 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & \text{for } n=1 \\ \mathbf{Z}/2 & \text{for } n=2 \\ 0 & \text{otherwise} \end{cases}$$

$$\bigoplus_{i+j=n-1} \mathrm{Tor}\,(H_i(X;\mathbf{Z}), H_j(X;\mathbf{Z})) = \begin{cases} \mathbf{Z}/2 & \text{for } n=3 \\ 0 & \text{otherwise} \end{cases}$$

so

$$H_n(X^2;\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } n=0 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & \text{for } n=1 \\ \mathbf{Z}/2 & \text{for } n=2 \\ \mathbf{Z}/2 & \text{for } n=3 \\ 0 & \text{otherwise} \end{cases}$$

(c) (5 POINTS) Find $H_*(X^3;\mathbf{Z}/2)$.

Solution: A similar calculation to (a) gives

$$H_n(X^3;\mathbf{Z}/2) = \bigoplus_{i+j=n} H_i(X;\mathbf{Z}/2) \otimes H_j(X^2;\mathbf{Z}/2)$$

$$= \begin{cases} \mathbf{Z}/2 & \text{for } n=0 \\ (\mathbf{Z}/2)^3 & \text{for } n=1 \\ (\mathbf{Z}/2)^6 & \text{for } n=2 \\ (\mathbf{Z}/2)^7 & \text{for } n=3 \\ (\mathbf{Z}/2)^6 & \text{for } n=4 \\ (\mathbf{Z}/2)^3 & \text{for } n=5 \\ \mathbf{Z}/2 & \text{for } n=6 \\ 0 & \text{otherwise} \end{cases}$$

(d) (5 POINTS) For a space Y define the mod 2 Poincaré series to be

$$g_{\mathbf{Z}/2}(Y)(t) = \sum_{n\geq 0} \mathrm{rank}\,(H_n(Y;\mathbf{Z}/2))t^n.$$

Find $g_{\mathbf{Z}/2}(X^k)(t)$ for $k \geq 0$.

Solution: Since mod 2 homology converts Cartesian products to tensor products,

$$g_{\mathbf{Z}/2}(A \times B)(t) = g_{\mathbf{Z}/2}(A)(t) g_{\mathbf{Z}/2}(B)(t).$$

Since $g_{\mathbf{Z}/2}(X)(t) = 1 + t + t^2$,

$$g_{\mathbf{Z}/2}(X^k)(t) = (1 + t + t^2)^k.$$

3. **Covering space question.** Let $\zeta = e^{2\pi i/3}$, let $\tilde{X}$ be the complement of the set

$$\{z_0 = 0, z_1 = 1, z_2 = \zeta, z_3 = \zeta^2\}$$

in $\mathbf{C}$, and let X be the complement of the set $\{0, 1\}$ in $\mathbf{C}$. Let $p : \tilde{X} \to X$ be defined by $p(z) = z^3$. Using the point $\tilde{x}_0 = 1/2 \in \tilde{X}$ as a base point, we define four closed paths ω_k for $0 \le k \le 3$ in $\tilde{X}$ as follows:

$$\begin{aligned}
\omega_0(t) &= e^{2\pi i t}/2 && \text{for } 0 \le t \le 1 \\
\omega_1(t) &= 1 - (e^{2\pi i t}/2) && \text{for } 0 \le t \le 1 \\
\omega_2(t) &= \begin{cases} e^{2\pi i t}/2 & \text{for } 0 \le t \le 1/3 \\ \zeta(1 - (e^{6\pi i t}/2)) & \text{for } 1/3 \le t \le 2/3 \\ e^{-2\pi i t}/2 & \text{for } 2/3 \le t \le 1 \end{cases} \\
\omega_3(t) &= \begin{cases} e^{-2\pi i t}/2 & \text{for } 0 \le t \le 1/3 \\ \zeta^2(1 - (e^{6\pi i t}/2)) & \text{for } 1/3 \le t \le 2/3 \\ e^{2\pi i t}/2 & \text{for } 2/3 \le t \le 1 \end{cases}
\end{aligned}$$

(I suggest you draw a picture of these paths.)

(a) (5 POINTS) Find $\pi_1(\tilde{X}, \tilde{x}_0)$ and describe the elements in it represented by the 4 closed paths ω_k.

Solution: Since $\tilde{X}$ is the complement of 4 points in the plane, its π_1 is the free group on 4 generators, say a_k for $0 \le k \le 3$. The four paths each go around one of them in a counterclockwise direction, so each ω_k represents one of the generators a_k.

(b) (5 POINTS) Show that p is a 3-sheeted covering.

Solution: The preimage of every every point in X is a set of three points in $\tilde{X}$.

(c) (5 POINTS) Let $x_0 = p(\tilde{x}_0) \in X$ and find $\pi_1(X, x_0)$. Describe the elements in it represented by the 4 closed paths $p\omega_k$. You may assume that the image under p of a circle of radius 1/2 about a cube root of unity is a simple closed curve going counterclockwise around 1 and not going around 0.

> **Solution:** Since X is the complement of 2 points in the plane, its π_1 is the free group on 2 generators, say x and y corresponding to 0 and 1. Then drawing suitable pictures shows that
>
> $$p(a_0) = x^3$$
> $$p(a_1) = y$$
> $$p(a_2) = xyx^{-1}$$
> $$p(a_3) = x^{-1}yx$$

(d) (5 POINTS) Find a homomorphism $\varphi : \pi_1(X, x_0) \to C_3$ whose kernel contains $p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

> **Solution:** Let $\gamma \in C_3$ be a generator, and define φ by $\varphi(x) = \gamma$ and $\varphi(y) = e$.

4. **Euler characteristic question.** (20 POINTS) Let X be a graph with V vertices and E edges. Embed it in $\mathbf{R}^3$ (there is a theorem saying that any graph can be embedded in 3-space; there are some that cannot be embedded in the plane) and let Y be the space of all points within ϵ (a sufficiently small positive number) of the image of X. It is a 3-manifold bounded by a surface M. Find the Euler charcterisitic $\chi(M)$ and prove your answer.

   HINT: Think of the building set in the lounge, the one with steel balls and black magnetic rods. We are going to build something with V balls and E rods. Find the Euler characteristic of the set of V 2-spheres bounding the V balls. Think about how the Euler characteristic of the surface changes each time you add a rod. *You may use the fact that*

   $$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

   under suitable hypotheses on A and B.

> **Solution:** The Euler characteristic of the disjoint union of V 2-spheres is $2V$. When we add an edge to the graph, we remove a disk from each of two (not necessarily distinct) spheres. This reduces χ by two. We then add a cyclinder by gluing its two boundary components to the two circles created by removing the two disks. This does not change χ, because both the cylinder and its boundary components have Euler characteristic zero. We do this E times, so $\chi(M) = 2V - 2E$.

5. **Last question.** (20 POINTS) Let X_1 be the 1-skeleton of a cube, which is a graph with 8 vertices and 12 edges. Let X_2 be the 1-skeleton of a tetrahedron, which is a graph with 4 vertices and 6 edges. Let M_1 and M_2 be the two corresponding surfaces as in the previous problem. Construct maps $X_1 \to X_2$ and $M_1 \to M_2$ which are double coverings.

Solution: Embed X_1 in $\mathbf{R}^3$ as the edges of the unit cube centered at the origin, with vertices at the points $(\pm 1/2, \pm 1/2, \pm 1/2)$. The group $G = C_2$ acts freely on the complement of the origin (which is homeomorphic to $S^2 \times \mathbf{R}$) by sending (x, y, z) to $(-x, -y, -z)$. The orbit space is $\mathbf{R}P^2 \times \mathbf{R}$. This action preserves the image of X_1 and its bounding surface M_1. The orbit space X_1/G is a graph with 4 vertices and 6 edges, half the number in X_1. Like X_1 it has three edges meeting at each vertex, so it is homeomorphic to X_2. It follows that M_1/G is homeomorphic to M_2. The desired double coverings are the maps of X_1 and M_1 to their orbit spaces.