

J is **final** in K . A **cofinal** or **initial functor** $J \rightarrow K$ is one that induces a final functor $J^{op} \rightarrow K^{op}$.

For more details, see [KS06, §2.5], where the term “co-cofinal” is used for final.

The nonemptiness of $(k \downarrow \alpha)$ means that for each object k in K there is an object j in J such that there is a morphism $k \rightarrow \alpha(j)$. Its connectivity means that for any two such j s there is a finite commutative diagram in K of the form

$$\begin{array}{c}
 k \\
 \swarrow \quad \downarrow \quad \searrow \\
 \alpha(j_0) \longrightarrow \cdots \longleftarrow \cdots \longrightarrow \cdots \longleftarrow \cdots \longrightarrow \alpha(j_n)
 \end{array}
 \tag{2.3.81}$$

where the morphisms in the bottom row are in the image of α , and the left and right morphisms from k are given.

The following was proved by Mac Lane as [ML98, Theorem IX.3.1].

Theorem 2.3.82. Colimit maps induced by final functors. *For a final functor $\alpha : J \rightarrow K$ as in Definition 2.3.80, if $X : K \rightarrow \mathcal{C}$ is a functor for which $\text{colim}_J X\alpha$ exists, then $\text{colim}_K X$ also exists and the induced map ϕ_α of (2.3.79) is an isomorphism.*

Corollary 2.3.83. Colimits indexed by categories with terminal objects. *Suppose the small category K has a terminal object k as in Example 2.1.16(ii) and $X : K \rightarrow \mathcal{C}$ is a functor. Then $\text{colim}_K X$ exists and is equal to the value of X on k .*

Proof Let J be the trivial category and let $\alpha : J \rightarrow K$ send its one object to k . This functor is easily seen to be final as in Definition 2.3.80, so the result is a special case of Theorem 2.3.82. \square

2.4 Ends and coends

Yoneda originally introduced ends and coends in the context of functors enriched (see §3.1 below) over $\mathcal{A}b$ in [Yon60, §4, page 545]. He called them the “integration” and “cointegration” and denoted them by

$$\int_J H \quad \text{and} \quad \int_J^* H$$

or a functor $H : J^{op} \times J \rightarrow \mathcal{C}$ from a small category J to a complete or cocomplete category \mathcal{C} . In this book we will denote the end and coend by

$$\int^J H \quad \text{and} \quad \int_J H$$

respectively. **We will use a superscript for an end and a subscript for a coend.** This differs from the notation of [ML71, pages 222–223] and most other works in category theory, where the opposite convention is used. However it agrees with the notation used for coends by Jacob Lurie in [Lur09, Chapter 2 and Appendix A], and in some papers on factorization homology such as [AF19].

Thus H is a functor of two variables in J , contravariant in the first and covariant in the second. For example we could have $\mathcal{C} = \mathit{Set}$ and $H(j, j') := J(j, j')$, the set of morphisms $j \rightarrow j'$.

Given such a functor H , for each morphism $f : j \rightarrow j'$ in J we have a diagram in \mathcal{C} ,

$$\begin{array}{ccc} & & H(j, j) \\ & & \downarrow f_* \\ H(j', j') & \xrightarrow{f^*} & H(j, j'). \end{array}$$

which has a limit (the pullback) when \mathcal{C} is complete. We use the Yoneda's symbol

$$\int^J H(j, j),$$

now called an **end**, to denote the limit obtained by considering such diagrams for **all** morphisms f in J , assuming that the target category is complete. More explicitly, for each morphism $f \in \mathit{Arr} J$ we get a morphism

$$H(\mathit{Dom} f, \mathit{Dom} f) \xrightarrow{f_*} H(\mathit{Dom} f, \mathit{Cod} f)$$

in \mathcal{C} . Hence we get a morphism to the product of such sets over all f having domain j ,

$$H(j, j) \xrightarrow{\phi_*} \prod_{\substack{f \in \mathit{Arr} J \\ \mathit{Dom} f = j}} H(j, \mathit{Cod} f).$$

given by $p_f \phi_* = f_*$, where p_f denotes the projection of the product onto the factor corresponding to f . Now we take the product of these morphisms over all objects j in J and get

$$\prod_{j \in \mathit{Obj} J} H(j, j) \xrightarrow{\phi_*} \prod_{f \in \mathit{Arr} J} H(\mathit{Dom} f, \mathit{Cod} f). \quad (2.4.1)$$

In a similar fashion the morphism

$$H(\mathit{Cod} f, \mathit{Cod} f) \xrightarrow{f^*} H(\mathit{Dom} f, \mathit{Cod} f)$$

leads to

$$\prod_{j \in \text{Ob } J} H(j, j) \xrightarrow{\phi^*} \prod_{f \in \text{Arr } J} H(\text{Dom } f, \text{Cod } f). \quad (2.4.2)$$

In other words, we have the following diagram in which the products on the right are over all objects or all morphisms in J .

$$\begin{array}{ccc} H(j, j) & & \prod_j H(j, j) \\ \downarrow f_* & \rightsquigarrow & \downarrow \phi_* \\ H(j', j') \xrightarrow{f^*} H(j, j') & & \prod_{j': j \rightarrow j'} H(j', j') \xrightarrow{\phi^*} \prod_{f: j \rightarrow j'} H(j, j'). \end{array} \quad (2.4.3)$$

Dually, when \mathcal{C} is cocomplete, we have a similar diagram with coproducts over all objects or all morphisms in J .

$$\begin{array}{ccc} H(j', j) \xrightarrow{f_*} H(j', j') & & \coprod_{f: j \rightarrow j'} H(j', j) \xrightarrow{\varphi_*} \coprod_{j'} H(j', j') \\ \downarrow f^* & \rightsquigarrow & \downarrow \varphi^* \\ H(j, j) & & \coprod_j H(j, j) \end{array} \quad (2.4.4)$$

Definition 2.4.5. For a functor $H : J^{op} \times J \rightarrow \mathcal{C}$ for a small category J to a complete category \mathcal{C} , the **end**

$$\int^J H(j, j)$$

is the equalizer of

$$\int^J H(j, j) \dashrightarrow \prod_{j \in \text{Ob } J} H(j, j) \begin{array}{c} \xrightarrow{\phi^*} \\ \xleftarrow{\phi_*} \end{array} \prod_{f \in \text{Arr } J} H(\text{Dom } f, \text{Cod } f).$$

for ϕ_* and ϕ^* as in (2.4.1) and (2.4.2).

For a similar functor to a cocomplete category \mathcal{C} , the **coend**

$$\int_J H(j, j)$$

is the coequalizer of

$$\prod_{f \in \text{Arr } J} H(\text{Cod } f, \text{Dom } f) \begin{array}{c} \xrightarrow{\varphi_*} \\ \xleftarrow{\varphi^*} \end{array} \prod_{j \in \text{Ob } J} H(j, j) \dashrightarrow \int_J H(j, j), \quad (2.4.6)$$

with φ_* and φ^* as in (2.4.4).

In both cases the “variable of integration” j appears twice in the “integrand” and could be replaced by any other symbol for an object in J .

Alternatively, for each morphism $f : j \rightarrow j'$ in J , we have a diagram in \mathcal{C} ,

$$\begin{array}{ccc} H(j', j) & \xrightarrow{f^*} & H(j, j) \\ f_* \downarrow & & \downarrow f_* \\ H(j', j') & \xrightarrow{f^*} & H(j, j'). \end{array}$$

Suppose for the moment that \mathcal{C} is bicomplete. For a fixed pair of objects (j, j') in J we could combine the above for all morphisms $j \rightarrow j'$ and get

$$\begin{array}{ccc} \coprod_{J(j, j')} H(j', j) & \xrightarrow{\varphi^*} & H(j, j) \\ \varphi_* \downarrow & & \downarrow \phi_* \\ H(j', j') & \xrightarrow{\phi^*} & \prod_{J(j, j')} H(j, j'). \end{array} \quad (2.4.7)$$

For cocomplete \mathcal{C} this leads to a coequalizer diagram

$$\coprod_{f \in \text{Arr } J} H(j', j) \begin{array}{c} \xrightarrow{\varphi^*} \\ \xrightarrow{\varphi_*} \end{array} \coprod_{k \in \text{Ob } J} H(k, k) \dashrightarrow \int_J H(k, k),$$

and for complete \mathcal{C} we have an equalizer diagram

$$\int^J H(k, k) \dashrightarrow \prod_{k \in \text{Ob } J} H(k, k) \begin{array}{c} \xrightarrow{\phi^*} \\ \xrightarrow{\phi_*} \end{array} \prod_{f \in \text{Arr } J} H(j, j').$$

Proposition 2.4.8. Ends and coends on the walking arrow category.

Let J be walking arrow category $(0 \rightarrow 1)$ as in [Definition 2.1.6](#), let \mathcal{C} be a cocomplete category and let $H : J^{op} \times J \rightarrow \mathcal{C}$ be a functor. Then

$$\int_J H(j, j) \cong H(0, 0) \amalg_{H(1, 0)} H(1, 1),$$

the pushout of the diagram

$$\begin{array}{ccc} & H(1, 0) & \\ \alpha^* \swarrow & & \searrow \alpha_* \\ H(0, 0) & & H(1, 1), \end{array} \quad (2.4.9)$$

where $\alpha : 0 \rightarrow 1$ denotes the unique nonidentity morphism in J .

Dually, for complete \mathcal{C} ,

$$\int^J H(c, c) \cong H(0, 0) \times_{H(0, 1)} H(1, 1),$$

the pullback of the diagram

$$\begin{array}{ccc} H(0,0) & & H(1,1) \\ & \searrow \alpha_* & \swarrow \alpha_* \\ & H(0,1) & \end{array}$$

Proof The diagram of (2.4.6) is

$$\begin{array}{c} H(0,0) \amalg H(1,0) \amalg H(1,1) \\ \varphi_* \downarrow \downarrow \varphi_* \\ H(0,0) \amalg H(1,1) \\ \downarrow \\ \int_J H(j,j). \end{array}$$

The restrictions of both φ^* and φ_* to $H(0,0)$ send it identically to $H(0,0)$, and similarly for their restrictions to $H_{1,1}$. This means that they contribute nothing to the coend, which is therefore the pushout of (2.4.9).

The dual case is similar. □

For a related result, see [Proposition 2.4.18](#) below.

The following are immediate consequences of the definitions.

Proposition 2.4.10. Functoriality of ends and coends. *Given two functors $H, H' : J^{op} \times J \rightarrow \mathcal{C}$, a natural transformation $\theta : H \Rightarrow H'$ induces morphisms*

$$\int_J \theta : \int_J H \rightarrow \int_J H' \quad \text{and} \quad \int^J \theta : \int^J H \rightarrow \int^J H'$$

with composition of natural transformations inducing composition of such morphisms.

Proposition 2.4.11. Limits (colimits) as ends (coends). *When the functor H is constant on the first variable, then its end (coend) is the usual limit (colimit) of H as a functor of the second variable for complete (cocomplete) \mathcal{C} .*

Remark 2.4.12. Ends (coends) as limits (colimits). *Every end (coend) is a limit (colimit) since it an equalizer (coequalizer) by definition. The statement at hand concerns the case when an end (coend) over a small category J is also an ordinary limit (colimit) over J .*

Proof This follows from the definitions and the calculation of [Example 2.3.35\(iii\)](#). □

Given a functor $H : J^{op} \times J \rightarrow \mathcal{C}$ and objects X and Y in \mathcal{C} , there are *Set*-valued functors on $J^{op} \times J$,

$$J^{op} \times J \xrightarrow{t} J \times J^{op} \xrightarrow{H^{op}} \mathcal{C}^{op} \xrightarrow{\mathcal{C}(-, Y)} \mathbf{Set} \quad (2.4.13)$$

and

$$J \times J^{op} \xrightarrow{H} \mathcal{C} \xrightarrow{\mathcal{C}(X, -)} \mathbf{Set}. \quad (2.4.14)$$

The following is immediate from the definitions.

Proposition 2.4.15. End/coend duality. *Given a functor H from $J^{op} \times J$ (for a small category J) to a cocomplete category \mathcal{C} , and an object Y in \mathcal{C} , there is a natural isomorphism*

$$\mathcal{C} \left(\int_J H, Y \right) \cong \int^J \mathcal{C}(H, Y),$$

where the expression on the left is the set of morphisms from the indicated coend to Y , and the expression on the right is the end of the *Set*-valued functor of (2.4.13).

For an object X in \mathcal{C} , there is a natural isomorphism

$$\mathcal{C} \left(X, \int^J H \right) \cong \int^J \mathcal{C}(X, H),$$

where the expression on the left is the set of morphisms from X to the indicated end, and that on the right is the end for the functor of (2.4.14).

An enriched version of the above is [Proposition 3.2.16](#) below.

There is a converse to [Proposition 2.4.11](#). It is taken from [\[ML98, IX.5\]](#) where it is stated for ends and limits. We will construct a new small category J_{\S} (Mac Lane's notation for the opposite category is J^{\S}) such that the coend of [Definition 2.4.5](#) is the colimit of a certain \mathcal{C} -valued functor on J_{\S} .

Definition 2.4.16. The cosubdivision category of a small category.

For a small category J , let J_{\S} be the category whose objects are symbols j_{\S} and f_{\S} for objects j and arrows f in J . Note that j_{\S} and $(1_j)_{\S}$ are different objects. The only nonidentity morphisms are arrows

$$j_{\S} \leftarrow f_{\S} \rightarrow j'_{\S}$$

for each arrow $f : j \rightarrow j'$ in J .

Given a functor $H : J^{op} \times J \rightarrow \mathcal{C}$, let $H_{\S} : J_{\S} \rightarrow \mathcal{C}$ be the functor indicated by the following diagram.

$$\begin{array}{ccccc} j_{\S} & \longleftarrow & f_{\S} & \longrightarrow & j'_{\S} \\ \downarrow & & \downarrow & & \downarrow \\ H(j, j) & \xleftarrow{f^*} & H(j', j) & \xrightarrow{f_*} & H(j', j') \end{array}$$

Dually, let the **subdivision category of J** be $J^\S = (J_\S)^{op}$. We denote the corresponding objects in it by j^\S and f^\S , and the only nonidentity morphisms are arrows

$$j^\S \rightarrow f^\S \leftarrow (j')^\S$$

for each arrow $f : j \rightarrow j'$ in J . The functor $H^\S : J^\S \rightarrow \mathcal{C}$ is indicated by

$$\begin{array}{ccccc} j^\S & \xrightarrow{\quad} & f^\S & \xleftarrow{\quad} & (j')^\S \\ \downarrow & & \downarrow & & \downarrow \\ H(j, j) & \xrightarrow{\quad f_* \quad} & H(j, j') & \xleftarrow{\quad f_* \quad} & H(j', j'). \end{array}$$

The following is stated for coends only. Its proof and that of its dual can be found in [ML98, IX.8]

Proposition 2.4.17. Fubini theorem for coends. *Let*

$$H : J_1^{op} \times J_2^{op} \times J_1 \times J_2 \rightarrow \mathcal{C}$$

for small categories J_1 and J_2 and a cocomplete category \mathcal{C} . Then for any pair $(a, b) \in J_1^{op} \times J_1$, we have the functor

$$H(a, -, b, -) : J_2^{op} \times J_2 \rightarrow \mathcal{C},$$

and its coend

$$\int_{J_2} H(a, c, b, c)$$

is a functor on $J_1^{op} \times J_1$, so the double coend

$$\int_{J_1} \int_{J_2} H(a, c, a, c)$$

is defined. Similarly we can define the double coend

$$\int_{J_2} \int_{J_1} H(a, c, a, c).$$

We can also define the coend on the product category

$$\int_{J_1 \times J_2} H(a, c, a, c).$$

These three objects in \mathcal{C} are naturally isomorphic.

Note that if the functor H above is constant on the contravariant variables, then Proposition 2.4.17 reduces to the statement that colimits over different diagrams commute with each other. The corresponding result about ends reduces to the commuting of limits.

The following is the double coend version of Proposition 2.4.8.

Proposition 2.4.18. Double coends on the walking arrow category.

Let J_1 and J_2 each be the walking arrow category $J = (0 \rightarrow 1)$ of [Definition 2.1.6](#), and let

$$H : J_1^{op} \times J_2^{op} \times J_1 \times J_2 \rightarrow \mathcal{C}$$

be a functor to a cocomplete category \mathcal{C} .

For each $(a, b) \in J^{op} \times J$, let

$$P(a, b) = \int_{c \in J} H(a, c, b, c),$$

which was identified as a certain pushout in [Proposition 2.4.8](#). Then

$$\int_{J \times J} H(a, c, a, c) \cong \int_J P(a, a) \cong P(0, 0) \amalg_{P(1, 0)} P(1, 1).$$

The following are special cases.

- (i) When the value of H is nontrivial (meaning not equal to \emptyset) only when both contravariant variables are 0, then the double coend is $H(0, 0, 0, 0)$.
- (ii) When the value of H is trivial when both contravariant variables are 1, then the double coend is the pushout of the diagram

$$\begin{array}{ccccc}
 & H(0, 1, 0, 0) & & H(1, 0, 0, 0) & \\
 & \swarrow & & \swarrow & \\
 H(0, 1, 0, \alpha) & & H(0, \alpha, 0, 0) & & H(\alpha, 0, 0, 0) \\
 \swarrow & & \swarrow & & \swarrow \\
 H(0, 1, 0, 1) & & H(0, 0, 0, 0) & & H(1, 0, 1, 0)
 \end{array}$$

- (iii) When the functor H is independent of the contravariant variables, then the double coend is $H(-, -, 1, 1)$.

Proof Using [Proposition 2.4.17](#), we have

$$\begin{aligned}
 \int_{(a, c) \in J \times J} H(a, c, a, c) &\cong \int_{a \in J} \int_{c \in J} H(a, c, a, c) \\
 &\cong \int_{a \in J} P(a, a) \\
 &\cong P(0, 0) \amalg_{P(1, 0)} P(1, 1).
 \end{aligned}$$

For (i),

$$\begin{aligned}
 \int_J H(0, 0, b, 0) &= H(0, 0, 0, 0) \amalg_{H(0, 0, 1, 0)} \emptyset \\
 &= H(0, 0, 0, 0)
 \end{aligned}$$

so

$$\int_{J \times J} H(a, c, a, c) = H(0, 0, 0, 0).$$

For (ii), since

$$P(a, b) = H(a, 0, b, 0) \amalg_{H(a,1,b,0)} H(a, 1, b, 1),$$

we have

$$\begin{aligned} P(1, 1) &= H(1, 0, 1, 0) \amalg_{H(1,1,1,0)} H(1, 1, 1, 1) \\ &= H(1, 0, 1, 0) \amalg_{\emptyset} \emptyset = H(1, 0, 1, 0), \\ P(1, 0) &= H(1, 0, 0, 0) \amalg_{H(1,1,0,0)} H(1, 1, 0, 1) = H(1, 0, 0, 0) \\ \text{and } P(0, 0) &= H(0, 0, 0, 0) \amalg_{H(0,1,0,0)} H(0, 1, 0, 1) \\ &= H(0, 1, 0, 1) \amalg_{H(0,1,0,0)} H(0, 0, 0, 0). \end{aligned}$$

It follows that the double coend is

$$\begin{aligned} P(0, 0) \amalg_{P(1,0)} P(1, 1) \\ &= \left(H(0, 1, 0, 1) \amalg_{H(0,1,0,0)} H(0, 0, 0, 0) \right) \amalg_{H(1,0,0,0)} H(1, 0, 1, 0) \\ &= H(0, 1, 0, 1) \amalg_{H(0,1,0,0)} H(0, 0, 0, 0) \amalg_{H(1,0,0,0)} H(1, 0, 1, 0), \end{aligned}$$

which is the indicated pushout.

For (iii), when the functor H of Proposition 2.4.17 is independent of the contravariant variables, the coend is an ordinary colimit by Proposition 2.4.11. Since $J \times J$ has terminal object $(1, 1)$, the coend in this case is $H(-, -, 1, 1)$. \square

Proposition 2.4.19. The set of natural transformations as an end.

Suppose we have two functors $F, G : J \rightarrow \mathcal{E}$ where J is small and \mathcal{E} is complete. Let $H : J^{op} \times J \rightarrow \mathcal{S}et$ be

$$H(C, C') = \mathcal{E}(F(C), G(C')).$$

Then the end

$$\int^J H(C, C) = \int^J \mathcal{E}(F(C), G(C))$$

is the set of natural transformations from F to G ,

$$Nat(F, G) = [J, \mathcal{E}](F, G).$$

Proof By Definition 2.4.5 the end is the equalizer of two morphisms from the product

$$\prod_{X \in J} \mathcal{E}(F(X), G(X)).$$

A natural transformation $\theta : F \rightarrow G$ assigns to each object X of J a morphism $\theta_X \in \mathcal{E}(F(X), G(X))$, so θ defines an element in the same product. The requirement that the diagrams (2.2.2) all commute is equivalent to requiring this element to be in the equalizer. \square

When $\mathcal{E} = \text{Set}$ and $F = \mathfrak{y}^A$, Proposition 2.4.19 reads

$$\int^{B \in J} \text{Set}(\mathfrak{y}^A(B), G(B)) = \int^{B \in J} \text{Set}(J(A, B), G(B)) = \text{Nat}(\mathfrak{y}^A, G).$$

The right hand side is $G(A)$ by the Yoneda Lemma 2.2.10, so we have the following.

Proposition 2.4.20. The Yoneda reduction. *Let J be a small category and $F : J \rightarrow \text{Set}$. Then for each object A of J ,*

$$\int^{B \in J} \text{Set}(J(A, B), F(B)) \cong F(A).$$

Now

$$\text{Set}(J(A, B), F(B)) = F(B)^{J(A, B)},$$

the Cartesian power of the set $F(B)$ indexed by the set $J(A, B)$. The right hand side is defined more generally for a functor F with valued in a complete category \mathcal{E} , and Proposition 2.4.20 has the following generalization.

Proposition 2.4.21. The generalized Yoneda reduction. *Let $F : J \rightarrow \mathcal{E}$ be a functor from a small category J to a complete category \mathcal{E} . Then for each object A of J ,*

$$\int^{B \in J} F(B)^{J(A, B)} \cong F(A).$$

Proof For each $f \in J(A, B)$ we get a map $F(f) : F(A) \rightarrow F(B)$. Collecting these for all f gives an evaluation map

$$i_B : F(A) \rightarrow F(B)^{J(A, B)}. \quad (2.4.22)$$

Collecting these for all objects B in the small category J defines a map

$$i : F(A) \longrightarrow \prod_{B \in J} F(B)^{J(A, B)}.$$

The end in question also supports a morphism to this product. It is by Definition 2.4.5 the equalizer of

$$\prod_{B \in \text{Ob } J} F(B)^{J(A, B)} \begin{array}{c} \xrightarrow{\phi^*} \\ \xrightarrow{\phi_*} \end{array} \prod_{h: B \rightarrow B'} F(B')^{J(A, B)}.$$

The equalizer is $F(A)$ because for each morphism $h : B \rightarrow B'$ in J , the following diagram commutes:

$$\begin{array}{ccccc}
 & & F(B')^{J(A,B')} & \xrightarrow{F(B')^{h*}} & F(B')^{J(A,B)} \\
 F(A) & \xrightarrow{i_{B'}} & & & \\
 & \searrow i_B & F(B)^{J(A,B)} & \xrightarrow{F(h)^{J(A,B)}} & F(B')^{J(A,B)}
 \end{array} \quad (2.4.23)$$

□

There is a dual formula for coends, which is sometimes called the **co-Yoneda lemma**. We will formulate and prove it simultaneously by dualizing the proof of [Proposition 2.4.21](#).

For a *Set*-valued functor F , map i_B of [\(2.4.22\)](#) is adjoint to

$$j_A : J(A, B) \times F(A) \rightarrow F(B).$$

The Cartesian product on the left, the disjoint union of copies of $F(A)$ indexed by the set $J(A, B)$, is defined whenever F takes values in a **cocomplete** category \mathcal{E} . We can take the coproduct of such things over all objects A of J and get a map

$$j : \coprod_{A \in J} J(A, B) \times F(A) \rightarrow F(B).$$

Then for each morphism $g : A' \rightarrow A$ in J , following diagram, which is dual to [\(2.4.23\)](#), commutes:

$$\begin{array}{ccccc}
 & & J(A', B) \times F(A') & \xleftarrow{g^* \times F(A')} & J(A, B) \times F(A') \\
 F(B) & \xleftarrow{j_{A'}} & & & \\
 & \searrow j_A & J(A, B) \times F(A) & \xleftarrow{J(A, B) \times F(g)} & J(A, B) \times F(A')
 \end{array}$$

This means that $F(B)$ can be described as a coend, and we have proved the following.

Proposition 2.4.24. The generalized Yoneda coreduction. *Let $F : J \rightarrow \mathcal{E}$ be a functor from small category J to a cocomplete category \mathcal{E} . Then for each object B of J ,*

$$\int_{A \in J} J(A, B) \times F(A) \cong F(B).$$

We will describe another approach to this for *Set*-valued functors below in [Example 2.5.14](#).

Remark 2.4.25. The case of a bicomplete category \mathcal{E} . By interchanging A and B , we can rewrite [Proposition 2.4.21](#) as

$$F(B) \cong \int^{A \in J} F(A)^{J(B,A)},$$

while [Proposition 2.4.24](#) gives

$$F(B) \cong \int_{A \in J} J(A, B) \times F(A).$$

Note that the first formula for $F(B)$ involves $J(B, A)$ while the second involves $J(A, B)$. It has to be this way because both expressions must be covariant in B , which $J(A, B)$ is. The expression in the end is contravariant in $J(B, A)$, which itself is contravariant in B .

2.5 Kan extensions

The notion of Kan extensions subsumes all the other fundamental concepts of category theory.

Saunders Mac Lane, [ML98, X.7]

2.5A Definitions and examples

Suppose we have functors F and K as in the diagram

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\
 & \searrow K & \downarrow \eta \\
 & & \mathcal{D} \\
 & & \nearrow L \\
 & & \mathcal{E}
 \end{array}
 \tag{2.5.1}$$

and we wish to extend the functor F along K to a new functor $L : \mathcal{D} \rightarrow \mathcal{E}$ with a natural transformation $\eta : F \Rightarrow LK$. We do **not** require L to be an actual extension of F , meaning we do not require that $LK = F$. We only require that the two be related by the natural transformation $\eta : F \Rightarrow LK$. We want it to have the following universal property: given another such extension G and natural transformation $\gamma : F \Rightarrow GK$

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\
 & \searrow K & \downarrow \gamma \\
 & & \mathcal{D} \\
 & & \nearrow G \\
 & & \mathcal{E}
 \end{array}$$