Math 443 **Final exam**

December 16, 2015

Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

RETURN THIS SHEET OF PAPER WITH YOUR BLUEBOOK.

1. **Euler characteristic question.** (20 POINTS) Let X be a finite graph with V vertices and E edges. Embed it in $\mathbf{R}^3$ (there is a theorem saying that any graph can be embedded in 3-space; there are some that cannot be embedded in the plane) and let Y be the space of all points within ϵ (a sufficiently small positive number) of the image of X. It is a 3-manifold bounded by a surface M. Find the Euler charcterisitic $\chi(M)$ and prove your answer.

   HINT: Think of the building set in the lounge, the one with steel balls and black magnetic rods. We are going to build something with V balls and E rods. Find the Euler characteristic of the set of V 2-spheres bounding the V balls. Think about how the Euler characteristic of the surface changes each time you add a rod. *You may use the fact that*

   $$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

   under suitable hypotheses on A and B.

   Solution: The Euler characteristic of the disjoint union of V 2-spheres is $2V$. When we add an edge to the graph, we remove a disk from each of two (not necessarily distinct) spheres. This reduces χ by two. We then add a cyclinder by gluing its two boundary components to the two circles created by removing the two disks. This does not change χ, because both the cylinder and its boundary components have Euler characteristic zero. We do this E times, so $\chi(M) = 2V - 2E$.

2. **Finite graph question.** (20 POINTS) Let X_1 be the 1-skeleton of an octahedron, which is a graph with 6 vertices and 12 edges. Let X_2 be a graph with 3 vertices and 2 edges connecting each pair of vertices, making 6 edges in all. Let M_1 and M_2 be the two corresponding surfaces as in the previous problem. Construct maps $X_1 \to X_2$ and $M_1 \to M_2$ which are double coverings.

   Solution: Embed X_1 in $\mathbf{R}^3$ as the edges of the octahedron centered at the origin, with vertices at the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. The group $G = C_2$ acts freely on the complement of the origin (which is homeomorphic to $S^2 \times \mathbf{R}$) by sending (x, y, z) to $(-x, -y, -z)$. The orbit space is $\mathbf{R}P^2 \times \mathbf{R}$. This action preserves the image of X_1 and its bounding surface M_1. The orbit space X_1/G is a graph with 3 vertices and 6 edges, half the number in X_1. Like X_1 it has four edges meeting at each vertex, and each edge has distinct endpoints. It follows that it is homeomorphic to X_2. It follows that M_1/G is homeomorphic to M_2. The desired double coverings are the maps of X_1 and M_1 to their orbit spaces.

3. **Infinite graph question.** (30 POINTS) Consider the infinite graph K in $\mathbf{R}^3$ with vertex set

$$\{(i,j,k)\in\mathbf{R}^3\colon i,j,k\in\mathbf{Z}\}\cup\left\{\left(\frac{2i+1}{2},\frac{2j+1}{2},\frac{2k+1}{2}\right)\in\mathbf{R}^3\colon i,j,k\in\mathbf{Z}\right\}$$

in which each vertex of the form (x,y,z) is connected by an edge to the eight neighboring vertices

$$\left\{\left(x\pm\frac{1}{2},y\pm\frac{1}{2},y\pm\frac{1}{2}\right)\right\}.$$

Thus the center of each edge is a point in the set

$$\left\{\left(i\pm\frac{1}{4},j\pm\frac{1}{4},k\pm\frac{1}{4}\right)\colon i,j,k\in\mathbf{Z}\right\}.$$

The two endpoints for such an edge with a given combination of signs are

$$(i,j,k)\qquad\text{and}\qquad\left(i\pm\frac{1}{2},j\pm\frac{1}{2},k\pm\frac{1}{2}\right)$$

with the same combination of signs in the second point. There is a corresponding surface M as in the previous two problems.

The group $G=\mathbf{Z}^3$ acts freely $\mathbf{R}^3$ by translation, with $(i,j,k)\in\mathbf{Z}^3$ sending $(x,y,z)\in\mathbf{R}^3$ to $(x+i,y+j,z+k)$. Hence it acts freely on both K and M. *Describe the finite orbit graph K/G and find the genus of the compact orbit surface M/G.* Both K/G and M/G are contained in the 3-dimensional torus $\mathbf{R}^3/G\cong S^1\times S^1\times S^1$, which is also a quotient of the unit cube.

Solution: The orbit graph has two vertices, the orbits of

$$(0,0,0)\qquad\text{and}\qquad\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

They are connected to each other by 8 edges, the orbits of the ones centered at the points

$$\left(\pm\frac{1}{4},\pm\frac{1}{4},\pm\frac{1}{4}\right).$$

hence $V=2$ and $E=8$. The result problem 1 implies that $\chi(M)=2V-2E=-12$, so the genus of M is 7.

Suppose we take the cube $[-1/2,1/2]^3$ as a fundamental domain for the group action on $\mathbf{R}^3$. Then the point $(0,0,0)$ is its center and each vertex maps to the orbit of $(1/2,1/2,1/2)$. The edges of K/G correspond to the 8 lines connecting the center of the cube to the cube's vertices.

4. (20 POINTS) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk D^2 to itself has a fixed point. You may assume $\pi_1 S^1 = \mathbf{Z}$.

> **Solution:** See page 32 of Hatcher.

5. (30 POINTS) Let M_g be a closed oriented surface of genus g. Its homology is as follows.

$$H_i(M_g) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ \mathbf{Z}^{2g} & \text{for } i = 1 \\ \mathbf{Z} & \text{for } i = 2 \\ 0 & \text{for } i > 2 \end{cases}$$

Let $M_{g,k}$ be M_g with k disjoint open disks removed. Compute $H_*(M_{g,k})$ for $k > 0$ and prove your answer.

Solution:

We use the Mayer-Vietoris sequence in which $A = M(g,k)$, B is k copies of D^2 and $C = A \cap B$ is k copies of S^1. Then we have

$$\begin{array}{ccccccc}
 & 0 & & H_2(M_{g,k}) & & \mathbf{Z} & \\
 & \| & & \| & & \| & \\
\cdots \longrightarrow & H_2(C) & \longrightarrow & H_2(A) \oplus H_2(B) & \longrightarrow & H_2(M_g) & \xrightarrow{\partial_2} \\
 & \mathbf{Z}^k & & H_1(M_{g,k}) & & \mathbf{Z}^{2g} & \\
 & \| & & \| & & \| & \\
\longrightarrow & H_1(C) & \longrightarrow & H_1(A) \oplus H_1(B) & \longrightarrow & H_1(M_g) & \xrightarrow{\partial_1} \\
 & \mathbf{Z}^k & & H_0(M_{g,k}) \oplus \mathbf{Z}^k & & \mathbf{Z} & \\
 & \| & & \| & & \| & \\
\longrightarrow & H_0(C) & \longrightarrow & H_0(A) \oplus H_0(B) & \longrightarrow & H_0(M_g) & \longrightarrow 0
\end{array}$$

Now $M_{g,k}$ is not a closed manifold, so $H_2(M_{g,k}) = 0$ and ∂_2 is one-to-one. It is path connected so $H_0(M_{g,k}) = \mathbf{Z}$ and $\partial_1 = 0$. It follows that

$$H_1(M_{g,k}) = \mathbf{Z}^{2g+k-1}.$$