THE STRUCTURE OF $SO(4)$
MATH 443
APRIL 2, 2002

This is the solution to the extra credit problem due on March 19, namely a proof that $SO(4)$ is homeomorphic to $SO(3) \times S^3$.

For general $n$ one has a map $p : SO(n) \to S^{n-1}$ obtained by evaluating an orthogonal matrix $g \in SO(n)$ on a fixed unit vector $x_0 \in S^{n-1}$. We claim that the preimage of each point under $p$ is homeomorphic to $SO(n-1)$. First note that $p^{-1}(x_0)$ is the set of matrices which fix the subspace spanned by $x_0$, so it is the group of special orthogonal matrices acting on the hyperplane orthogonal to $x_0$.

More generally two elements in $p^{-1}(x)$ differ by an element of $SO(n)$ that fixes $x$, i.e., by a special orthogonal matrix acting on the hyperplane orthogonal to $x$. If we pick an element $g_x \in p^{-1}(x)$, then for any $g \in p^{-1}(x)$, $g^{-1}g_x$ is in the subgroup of $G_x \subset SO(n)$ that fixes $x$, and $G_x$ is isomorphic to $SO(n-1)$.

Now suppose the map $p$ has a lifting, i.e., a map $\ell : S^{n-1} \to SO(n)$ such that $p \ell$ is the identity map on $S^{n-1}$. Then arguments similar to those of 6.14 show that $SO(n)$ is homeomorphic to $S^{n-1} \times SO(n-1)$. It turns out that $\ell$ exists only if $n = 2, 4$ or $8$.

The lifting for $n = 4$ is based on the structure of the quaternions $\mathbf{H}$ (see http://mathworld.wolfram.com/Quaternion.html). This a noncommutative division algebra homeomorphic to $\mathbb{R}^4$. Mutiplication by any nonzero unit vector induces a special orthogonal transformation of $\mathbb{R}^4$, so we get a group structure on $S^3$ and a homomorphism $S^3 \to SO(4)$, which is the desired lift.

The lifting for $n = 8$ follows in a similar way from the existence of the Cayley numbers (see http://mathworld.wolfram.com/CayleyAlgebra.html), a nonassociative division algebra homeomorphic to $\mathbb{R}^8$.

The nonexistence of liftings for other values of $n$ is a deeper theorem in algebraic topology. A good historical reference for it is May’s paper

http://www.math.uiuc.edu/K-theory/0321/

starting on page 15.