Recall facts about $\pi_1$.

1. $\pi_1(S^n) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$

$\pi_1 SO(2) = \pi_1(S^1) \cong \mathbb{Z}$

2. $SO(3)$ is the group of rotations in $\mathbb{R}^3$.

$\mathbb{R}P^3 = \text{space of lines through the origin in } \mathbb{R}^4$.

$\pi_1 SO(3) \cong \mathbb{Z}/2\mathbb{Z}$

3. $SO(2) \to SO(3)$ induces a map

$\pi_1 SO(2) \to \pi_1 SO(3)$

$\pi_1 SO(2)$ is generated by a closed path corresponding to a full
rotation: \( \theta \mapsto \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \)

\(0 \leq \theta \leq 2\pi\)

Double rotation \( \theta \mapsto \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \)

This closed path in \( SO(2) \)

represents a generator in \( \pi_1 \), \( SO(2) \)

\( SO(2) \xrightarrow{\text{onto}} SO(3) \)

\( \mathbb{Z} \xrightarrow{\text{onto}} \mathbb{Z}/2 \)

\( 2 \cdot \text{gen} \xrightarrow{\text{onto}} 0 \)

Def \( \mathbb{RP}^n = \text{space of lines through } 0 \)

in \( \mathbb{R}^{n+1} \)

is \( n \)-dimensional real projective space.
Equivalently, it is the quotient of $S^n$ obtained by identifying antipodal points.

$n = 1$

$\mathbb{R}P^3 \cong S^3$

It is also the quotient of $B^n$ (the $n$-dimensional ball) obtained by identifying antipodal points on the boundary.

$\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^3$

Experiment: Take a Möbius band

\[\text{Diagram of Möbius band}\]
Glue these together along their bounding circles and the surface obtained in $\mathbb{RP}^2$.

\[ \text{Bernard Motz} \]

Why is $\text{SO}(3)$ homeomorphic to $\mathbb{RP}^3$?

Every nontrivial rotation has an axis $S^2 \cong \mathbb{R}$ and an angle $\theta$ with $0 \leq \theta \leq \pi$.

So $(L, \theta)$ defines a point in $B^3$, a ball of radius $\pi$. If $\theta = \pi$, the counterclockwise rotations about $L$ and $-L$ are the same.

$\theta$ = distance from center of ball.
The map is onto but not 1-1. It is 1-1 on the interior of $B^3_T$ and 2-1 on the boundary.

Hence $SO(3) \cong \mathbb{RP}^3$.

Why does this mean $\chi_1 SO(3) = \frac{\pi}{2}$?
Def. A covering is a map \( \tilde{X} \to X \) s.t. each \( x \in X \) has a neighborhood \( U \) s.t. \( \tilde{p}^{-1}(U) \cong X \) is homeomorphic to \( U \times D \) where \( D \) is discrete, and the map \( \tilde{X} \to \tilde{p}^{-1}(U) \cong U \times D \) projection onto \( \tilde{X} = U \times \tilde{p} = \text{first co-ordinate} \).

Example 1) \( \tilde{X} = X \times D \) and \( p = \tilde{p}_1 \). This is called a trivial covering of \( X \).

2) \( \tilde{X} = \mathbb{R} \), \( X = \{ z \in \mathbb{C} : |z| = 1 \} \)

\[ \tilde{p}(z) = e^{it}, \quad \tilde{p}^{-1}(1) = \{ 2\pi n : n \in \mathbb{Z} \} \]

\( U \) could be any proper open subset.
If $S^1$, then $p^{-1}(U) = U \times \mathbb{Z}$.

3) $\mathbb{R}^n \rightarrow S^n \xrightarrow{p} \mathbb{R}P^n$

$x \mapsto \text{line through } x \text{ and } 0$.

$D$ has 2 points.

This is called a double covering.

Then for a covering $(\tilde{x}, \tilde{x}_0) \xrightarrow{p} (x_0, x_0)$, the homomorphism $\pi_1(\tilde{x}, \tilde{x}_0) \rightarrow \pi_1(x_0, x_0)$ is 1-1.