Def. Two spaces $X$ and $Y$ are homotopy equivalent if there are maps $f : X \to Y$ and $g : Y \to X$ such that:

$\oplus f \sim 1_X$ and $\oplus g \sim 1_Y$

Example: $X = \mathbb{R}^n$, $Y = \text{pt}$

$f = \text{constant map}$, $g(\text{pt}) = 0 \in \mathbb{R}^n$

$bg \sim 1_Y$ but $bf(x) = 0$ for all $x \in \mathbb{R}^n$

To show $gf \sim 1_X$, let
\[ h(x, t) = t \cdot x \quad \text{for} \quad 0 \leq t \leq 1 \]

\[ h(x, 0) = f \quad \text{and} \quad h(x, 1) = x \]

**Def.** A space \( X \) is **contractible** if it is homotopy equivalent to a point.

**Example.** \( \mathbb{R}^2 - \{0\} \) is homotopy equivalent to \( S^1 \).

\[ \mathbb{C} - \{0\} \quad \xrightarrow{\sim} \quad S^1 \]

\[ 2 \quad \xrightarrow{\sim} \quad \mathbb{Z} \]

**Def.** A null homotopy is one between a map \( f \) and a constant map. A map is **essential** if it is not null homotopic.

**Def.** A subspace \( A \hookrightarrow X \) is a
retract if there is a map \( X \xrightarrow{m} A \)
such \( m \circ i = 1_A \). \( m \) is a retraction.

Example

\[
A = S^1 \quad X = S^1 \times S^1 = \text{torus}
\]

\[-1 \xrightarrow{i} S^1 \quad \alpha \xrightarrow{m} S^1 \quad -1 \xrightarrow{1} (2, 1)
\]

\[
(2, 2) \xrightarrow{1} \mathbb{Z}
\]

INTOXICATION WARNING

Categories & functors

Def A category \( \mathcal{C} \) consists of
1) A class of objects
2) For each pair of objects \( X \) and \( Y \), a set of morphisms \( X \to Y \).
Each object $X$ has an identity morphism. Morphisms can be composed and composition is associative.

**Examples**

1. **Sets**
   - Objects are sets
   - Morphisms are maps

2. **Top**
   - Objects are top. spaces
   - Morphisms are continuous maps

3. **Ab**
   - Objects are abelian groups
   - Morphisms are group homomorphisms
Def: Given two categories \( \mathcal{C} \) and \( \mathcal{D} \), a function \( F: \mathcal{C} \to \mathcal{D} \) is a rule which assigns to each object \( X \) in \( \mathcal{C} \) an object \( F(X) \) in \( \mathcal{D} \), and to each morphism \( X \to Y \) in \( \mathcal{C} \) a morphism \( F(X) \to F(Y) \) in \( \mathcal{D} \), with \( F(\alpha \beta) = F(\alpha)F(\beta) \).

Examples:
1. Forgetful functors
   \[ \text{Top} \to \text{Sets} \leftarrow \text{Ab} \]
2. Set \( \text{Top} \), be the category of topological spaces with base point (based or pointed spaces). Morphisms...
preserve base points. Let $\Pi_1$ be the category of groups. Then $\Pi_1$ (the fundamental group) is a functor

$$\text{Top} \xrightarrow{\Pi_1} \text{Grp}$$

will define $\Pi_n (X, x_0)$ for $n > 1$. It is abelian.

Consider the set of homotopy classes of maps

$$\begin{aligned}
\left( S^n, x_0 \right) &\longrightarrow (X, x_0) \\
\left( I^n, 2I^n \right) &\longrightarrow (X, x_0)
\end{aligned}$$

$\text{I}^n = \left[ 0, 1 \right]^n$ - $n$-cube

$$\partial I^n = \text{boundary of } I^n = \{(s_1, s_2, \ldots, s_n) : 0 \leq s_i \leq 1\}$$
\[ \{(a_1, \ldots, a_n) \in I^n : a_i = 0 \text{ or } 1 \text{ for some } i\} \]

Given two such maps \( \alpha, \beta \), define

\[ \alpha \times \beta \quad \text{by} \]

\[ \alpha \times \beta (a_1, \ldots, a_n) = \begin{cases} \alpha (2a_1, a_2, \ldots, a_n) & \text{for } 0 \leq a_i \leq 1/2 \\ \beta (2a_1, a_2, \ldots, a_n) & \text{for } 1/2 \leq a_i \leq 1 \end{cases} \]

Claim the resulting \( \alpha \times \beta \) is abelian

for \( n = 2 \). (Generalization to larger \( n \)

will be obvious.)

Let \( \alpha, \beta : (\mathbb{I}^2, \partial \mathbb{I}^2) \to X \)

\[ (0, 2) \quad \text{and} \quad (1, 2) \]

\[ \begin{array}{c|c|c}
\begin{array}{c}
\alpha \times \beta \\
\hline
\end{array} & \begin{array}{c}
T = 0 \\
\hline
\end{array} & \begin{array}{c}
\Theta \times \alpha \\
\hline
\end{array} \\
\hline
\end{array} \]

\[ \theta = 1 \]
\[ \pi_2(\lambda, x_0) \text{ is abelian for any } \lambda. \]

\[ \text{Top}_0 \xrightarrow{\text{In}} \text{Aut} \quad n \geq 2. \]