Theorem 1.7 \( \pi_1(S^2) = \mathbb{Z} \)

This follows from

\[
\begin{aligned}
Y \times \mathbb{R}^2 & \rightarrow X \\
\pi \downarrow & \quad \downarrow p \\
Y \times I & \rightarrow X \\
\end{aligned}
\]

Last time: Each \( y \in Y \) has a nbhd \( N \) such that \( \tilde{F} \) can be constructed on \( N \times I \).

To show \( \tilde{F} \) is unique on \( \tilde{x}y \tilde{z} \times I \):

Assume we have 2 liftings \( \tilde{F} \) and \( \tilde{F}' \).

Choose a partition of \( I = [0,1] \)
\[ 0 = t_0 < t_1 < \cdots < t_m = 1 \]

\( F(y; t_i, t_{i+1}) \) is some \( U_i \subset X \)

Assume inductively that \( \bar{F} \) and \( \bar{F}' \)

agree on \( y \times [t_0, t_1] \). Then \( \bar{F}(y; t_i, t_{i+1}) \)
and \( \bar{F}'(y; t_i, t_{i+1}) \) both lie in the

same copy of \( U_i \) in \( X \). \( \bar{F} \) is \( \bar{F}' \) on this

copy of \( U_i \), so \( \bar{F} \) and \( \bar{F}' \) agree on

\( y \times [t_i, t_{i+1}] \). This is inductive step.

\( \bar{F} \) on \( N \times I \) is unique on \( y \times I \) for each \( y \in \mathbb{N} \)
and hence unique on all of \( N \times I \).

For 2 intersecting sides \( N \) and \( N' \) with

\( N \cap N' \neq \emptyset \), \( \bar{F} \) is unique on \( (N \cap N') \times I \)
and hence on $(N \cup N') \times I$.

It follows that $\pi$ is unique on the whole space $Y \times I$. QED

This completes the proof of 1.7, that

$\Pi_1 \leq \leq 2$.

Remark: Suppose we relax the definition of covering by not requiring $D$ to be discrete. Then $\tilde{X} \rightarrow X$ is said to be a fiber bundle with fiber $D$. Then we can construct $F$ as before, but not uniquely.
A fibration is any map \( X \rightarrow Y \) for which \( \mathcal{F} \) can always be constructed. More about this later.

Recall the **Fundamental Theorem of Algebra** (due to Gauss):

If \( p(z) \in \mathbb{C}[z] \) is not a constant,

then it has a complex root \( z \)

\[ p(z) = 0 \quad \Rightarrow \quad p(z) = (z - z_0) q(z) \]

**Topological Proof**

Let \( p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \), with \( a_n \neq 0 \).

Assume \( p(z) \) has no root. For \( m \geq 0 \)
Let \( f_m(s) = \frac{\phi(m e^{2\pi i s})}{\phi(m)} \in S^1 \) for \( 0 \leq s \leq 1 \).

This is a closed path on \( S^1 \) that varies continuously with \( m \).

It is constant if \( m = 0 \).

We will show that for \( m > 0 \), the path is homotopic to \( \omega_n : s \mapsto e^{2\pi i m s} \).

Choose \( M > |a_1| + |a_2| + \cdots + |a_n| \). Then for \( |z| = M \),

\[
|z^n| = |z|^{n-1} > (|a_1| + \cdots + |a_n|) M^{n-1} \\
\geq |a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n|
\]

Let \( \phi_x(z) = z^n + x (a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n) \) for \( 0 \leq x \leq 1 \),
Note \( \phi_0(z) = z^n \) and \( \phi_1(z) = \phi(z) \).
\( \phi_0(z) \) has no roots \( z \) with \( |z| > m \).
This means \( f_m \cong f_0 \) and \( f_m \cong w_m \).
Hence by the previous Theorem, \( m = 0 \), CONTRADICTION. QED