Brouwer Fixed Point Theorem (1910)

Any map \( D^2 \to D^2 \) has a \( p \) such that \( p \in D^2 \) with \( f(p) = p \).

Proof: Assume \( f \) as above.

Define a map \( g : D^2 \to S^1 \) as shown.

\[
S^1 \hookrightarrow D^2 \xrightarrow{g} S^1
\]

Hence \( g \) extends the identity map \( S^1 \).

Consider the homotopy \( h : S^1 \times I \to S^1 \)

\[
h(x^1, t) = g(tx^1)
\]
\[
\begin{align*}
(x, 1) & \rightarrow x \\
(x, 0) & \rightarrow g(x)
\end{align*}
\]

If \( g \) is a null homotopy for \( I_{S^1} \), \textit{CONTRACTION} becomes \( \pi_1(S^1) = \mathbb{Z} \) and \( I_{S^1} \) is essential, i.e. not homotopic to constant map. QED

We could prove a similar thing about \( D^n \) if we knew that \( S^{n-1} \rightarrow S^n \) is essential. We will see this later.

\[ \pi_n(S^{n-1}) = \mathbb{Z} \text{ for all } n > 0. \]

**Thm 1.10 (Borsuk-Ulam) Borsuk-Ulam** \( S^2 \rightarrow \mathbb{R}^2 \)

\[ \exists x \in S^2 \text{ with } g(x) = g(-x). \]

**Proof**: Assume \( g \) as above.
Let \( g(x) = \frac{f(x) - f(-x)}{2} \in S^{1} \)

I \( \eta \rightarrow S^{2} \xrightarrow{\mu} S^{1} \) and \( g(-x) = -g(x) \)

\( \eta(x) = (\cos 2\pi x, \sin 2\pi x, 0) \in S^{2} \subset \mathbb{R}^{3} \)

Path around equator.

\( \tilde{h} \rightarrow (\mathbb{R}, r_{0}) \) \quad \tilde{h}(x + \frac{1}{2}) = -\tilde{h}(x) \)

\( (1, 0) \rightarrow (S^{1}, r_{0}) \) for an odd integer \( q \).

\( \tilde{h}(1) = \tilde{h}(\frac{1}{2}) + q/2 = \tilde{h}(0) + q/2 + q/2 \)

\( = r_{0} + q \) with \( q \neq 0 \)

\( \tilde{h} \) defines a nontrivial elt \( \tilde{h} \in \pi_{1}(S^{1}) = \mathbb{Z} \)

Since \( g \) extends to the northern hemisphere, \( \tilde{h} \) must be null. \textbf{CONTRACTION.}

QED.
Generalization to higher dimensions is more complicated, but true. 

Cor (Ham Sandwich Theorem)

Given 3 compact subsets of $\mathbb{R}^3$, there is a plane which intersects each of them.

If: Each oriented plane has a unit normal vector corresponding to a bit in $S^2$. Each plane is parallel to one which intersects $K_i$.

For each bit on $S^2$ we have such a plane.

Define $S^2 \rightarrow \mathbb{R}^2$.
\[ K_i^\pm = \text{portion of } K_i \text{ above the plane} \]

\[ K_i^- = \text{part below} \]

\[ f(x) = \left( \text{vol}(K_2^+) - \text{vol}(K_2^-) \right) \cdot \text{vol}(K_i^+) - \text{vol}(K_i^-) \]

Assume that \( f(x) \neq (0,0) \) for all \( x \).

Note \( f(-x) = -f(x) \).

\[ g(x) = \frac{f(x)}{[f(x)]} \in S^1 \]

\[ g(-x) = -g(x) \]

Such a map cannot exist. QED.

Van Kampen Theorem.

\[ X = A \cup B, \quad A, B \quad \text{s.t.} \quad A \cap B \quad \text{is path connected} \]

\[ x_0 \in A \cap B. \quad \text{Suppose we know} \]
Abstract definition. A pullout in a category $\mathcal{C}$ for the diagram

\[
\begin{array}{ccc}
K & \xrightarrow{i} & P \\
\downarrow & & \downarrow \text{id}
\end{array}
\]

is an object $P$ with morphisms as shown.
with the following universal property.

For any \((X, i, j)\) as shown with \(id = j\beta\),

\[ \exists ! \ g : P \rightarrow X \text{ with } g \circ i = i \text{ and } g \circ j = j. \]