Another proof that \( \pi_n(T^2) = \mathbb{Z} \oplus \mathbb{Z} \)

\[
\text{Theorem: } \pi_n(X \times Y) = \pi_n(X) \oplus \pi_n(Y)
\]

for any \( n \geq 1 \).

\( T^2 = S^1 \times S^1 \) so \( \pi_1(T^2) = \pi_1(S^1) \oplus \pi_1(S^1) = \mathbb{Z} \oplus \mathbb{Z} \)

\[\text{Proof: We have maps } X \xrightarrow{1_X} X \times Y \xrightarrow{1_Y} X \]

Assume \( X \) and \( Y \) have base points \( x_0 \) and \( y_0 \). The map \( X \to X \times Y \) sends \( x \) to \((x, y_0)\). This implies \( \pi_n(X) \) is a direct summand of \( \pi_n(X \times Y) \). Similarly for \( \pi_n(Y) \). We can get maps...
\[ \pi_n(X) \oplus \pi_n(Y) \rightarrow \pi_n(X \times Y) \rightarrow \pi_n(X) \oplus \pi_n(Y) \]

Identity arrow

It suffices to show \( f \) is \( 1 \)-1. A map \( f \) to \( X \times Y \) is determined uniquely by \( p_1 f \) and \( p_2 f \).

\[ s^m \rightarrow X \xrightarrow{f} \rightarrow X \times Y \xrightarrow{p_1} X \]

\( \text{QED} \)

Another example of Van Kampen Theorem

Surface of genus 2.

\[ A = \text{red part} \]
\[ A \cong \mathbb{T}^2 \text{ - open disk} \]
\[ A \cong S^1 \vee S^1 \]

\[ B = \text{green part} \]
\[ B \cong S^1 \vee S^1 \]

\[ A \cap B = S^1 \]
\( F_2 = \{a_1, b_1\} \)

\[ \alpha \text{ (gen)} = a_1 b_1 a_1^{-1} b_1^{-1} \]

\[ \beta \text{ (gen)} = (a_2 b_2 a_2^{-1} b_2^{-1})^{-1} \]

\[ F_2 = \{a_2, b_2\} \]

\[ \Pi_1 M_2 = \left\{ a_1, a_2, b_1, b_2 : a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = e \right\} \]

= nonabelian gp

---

Van Kampen Theorem: Let \( X = A \cap B \) where \( A \), \( B \) and \( A \cap B \) are path connected + nonempty. Then \( \Pi_1(X) \) is the pushout of

\[ \alpha \to \Pi_1(A) \]

basepoint \( x_0 \in A \cap B \),

\[ \beta \to \Pi_1(B) \]
Proof: Let \( P \) be the fundamental group.

\[
\begin{align*}
\pi_1(A \cap B) &\xrightarrow{P} \pi_1(A) \xrightarrow{\alpha} \pi_1(X) \\
\pi_1(B) &\xrightarrow{\beta} \pi_1(X)
\end{align*}
\]

Want to show \( P \) is iso. To show it is onto.

Suppose we have \((I, \partial I) \to (X, x_0)\)

We can find a partition:

\[
0 = s_0 < s_1 < s_2 < \ldots < s_n = 1
\]

such that \( f \) sends \([s_{i-1}, s_i]\) to \( A \) or \( B \).

Since \( A \cap B \) is path connected, we can find a path \( g_i \) in it from \( x_0 \) to \( f(s_i) \) for \( 0 \leq i \leq n \).
Let $f_i = f|_{[s_{i-1}, s_i]}$

$g_i = \text{inverse of } g_i$.

Then $g_i^{-1}f_i \ast g_i = \tilde{f}_i$ is a closed path in $A \ast B$

and $f = f_1 \ast f_2 \ast \ldots \ast f_n \cong \tilde{f}_1 \ast f_2 \ast \ldots \ast f_n$

This means $\tilde{f}$ is onto.

Call this a factorization of $[f] \in \Pi_1 X$.

$[\tilde{f}_1] [\tilde{f}_2] \ldots [\tilde{f}_n]$ is a word in $\Pi_1(A) \ast \Pi_1(B)$.

Two such factorizations are equivalent if they are related by sequences of the following two moves:

i) $f_i [\tilde{f}_i]$ and $[\tilde{f}_{i+1}]$ are in the same gp.
replace it by \([f_1 \times f_1']\)

ii) If \(f_1 \in \pi_1(A) \oplus \pi_1(B)\) and \(f_1\) has image in \(A \cap B\), we can change between \(A\) and \(B\).

Let \(F = \pi_1(A) \times \pi_1(B)\). First move does not change an element in \(F\).

Let \(N \triangleleft F\) be the normal subgroup generated by the image of \(\pi_1(A \cap B)\). The second leaves the image in \(F/N = P\) unchanged.

To show it is 1-1 we need to show
that any 2 factorizations of \([f]\) are equivalent as defined above.

And hence \([f] = [\bar{f}_1] [\bar{f}_2] \ldots [\bar{f}_k] = [\bar{f}_1]' [\bar{f}_2]' \ldots [\bar{f}_k]'

This means \(f_1 \times f_2 \ldots \times f_k \sim f_1' \times f_2' \ldots f_k'

Will examine the homotopy between them next time.