Given a $\Delta$-complex, i.e., a space obtained by gluing simplices together, we get a chain complex and we can compute its homology.

**Defects:** This only works for spaces constructed in this way. Two different $\Delta$-complexes could be homeomorphic as spaces. Do they have the same $H_{\ast}$?

**Def:** A singular $n$-simplex for a space $X$ is a map $\sigma : \Delta^n \to X$. 
Let $C_n(X)$ be the free abelian gp generated by all such maps.

To make this a chain complex we need maps $C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$.

Let $\{v_0, v_1, \ldots, v_n\}$ be the vertices of $\Delta^n$.

$\Delta^{n-2} \xrightarrow{g_i} \Delta^{n-1} \xrightarrow{f_i} \Delta^n \xrightarrow{\delta} X$

Let $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma f_i \in C_{n-1}(X)$.

$\sigma \in C_n(X)$

Exercise: $\partial_{n-1} \partial_n = 0$.
Def $H_n(X)$ is the $n$th homology of this chain complex.

This is clearly a functor:

Spaces $\xrightarrow{\Delta}$ Chain complexes

Given $X \xrightarrow{f} Y$

$\Delta^n \xrightarrow{d} C_n(X) \xrightarrow{f^*} C_n(Y)$

The only example where $C_*(X)$ can be explicitly described: $X =$ point.

$C_n(X) :=$ free abelian gp gen'd by all count maps $\Delta^n \rightarrow X$
0 \leq C_0(x) \leq C_1(x) \leq C_2(x) \leq C_3(x) \leq \cdots

n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}

\text{and } H_n(x) = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}

\text{Thm } H_n(\text{pt.}) = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}

(\text{Dimension axiom})

\text{Prop } \text{Let } X = \bigsqcup X_\alpha \text{ where each } X_\alpha \text{ is a path connected component of } X.
Then \( H_\ast(X) = \bigoplus \bigoplus H_\ast(X_\alpha) \).

\[ C_\ast(X) = \bigoplus \bigoplus C_\ast(X_\alpha) \text{ because the image of any map } \sigma : \Delta^n \to X \text{ is contained in some } X_\alpha. \]

Result follows. (QED)

\[ \text{Prop 2.7: Let } X \text{ be path connected and nonempty. Then } \quad H_0(X) = \mathbb{Z}. \]

\[ \text{Pf: Consider the hom } \varepsilon : C_0(X) \to \mathbb{Z} \text{ (augmentation)} \]

\[ C_0(X) = \text{ free ab gr genl by } \sigma : \Delta^0 \to X \]
$C_0(X) \ni \sum n_i \cdot x_i \mapsto \sum n_i = 2 \quad x_i \in X$

Claim 1) $C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$

is 0 i.e. $\ker \varepsilon \supset \text{im} \partial_1$

2) $\text{im} \partial_1 \supset \ker \varepsilon$.

Hence $C_0(X) / \text{im} \partial_1 = \mathbb{Z}$

To show 1): given a singular 1-simplex

$\Delta^1 \xrightarrow{\sigma} X$

$[\nu_0, \nu_1]$

$\partial_1(\sigma) = \sigma(\nu_0) - \sigma(\nu_1) \in C_0(X)$
\[ \exists \alpha_1(0) = 3 \alpha (w_0) - 3 \alpha (w_i) = 1 - 1 = 0. \]

To prove 2), let \( \alpha = \sum_{i=1}^{N} n_i \chi_i \in C_0(X) \) with \( \sum_{i=1}^{N} n_i = 0 \), i.e. it is in kernel \( \mathfrak{E} \). Need to find \( \beta \in C_1(X) \) with \( \mathfrak{E}(\beta) = \alpha \).

Choose a base at \( x_0 \in X \) and a path \( \gamma_i \) from \( x_0 \) to \( x_i \) for each \( i \).

Let \( \beta = \sum_{i=1}^{N} n_i \gamma_i \in C_1(X) \)

\[ \mathfrak{E}(\beta) = \sum_{i=1}^{N} n_i \mathfrak{E}(\gamma_i) = -\sum_{i=1}^{N} n_i (x_0 - x_i) \]

\[ = \sum_{i=1}^{N} n_i \chi_i - \sum_{i=1}^{N} n_i \chi_0 \]

\[ = \alpha - 0 = \alpha \quad \text{QED} \]