Homotopy axiom: If \( f, g : X \to Y \) are homotopic, then \( H_* (f) = H_* (g) \).

**Def.** Given 2 chain maps \( f, g : C \to C' \), a chain homotopy \( P \) between them is a collection of maps \( P_n : C_n \to C'_{n+1} \) which commutes with boundary maps: 

\[
\begin{align*}
C_{m+1} & \xrightarrow{d_{m+1}} C_m \\
C_n & \xrightarrow{d_n} C_{n-1} \\
C'_{m+1} & \xrightarrow{d'_{m+1}} C'_m \\
C'_{n} & \xrightarrow{d'_n} C'_{n-1}
\end{align*}
\]

\[
\begin{align*}
P_{m+1} & \circ d_{m+1} = d'_{m+1} \circ P_m \\
P_m & \circ d_n = d'_n \circ P_{m-1}
\end{align*}
\]
with \( P_{m+1} d_m + d_{m+1} P_m = f_m - g_m \) \( \forall m \).

Lemma If \( P \) exist as above, then
\[
H_x(f) = H_x(g).
\]

Proof: Let \( x \in H_m(C) \) be mapped by a cycle \( x \in C_n \). Then
\[
f_m(x) - g_m(x) = (P_{m+1} d_m + d_{m+1} P_m)(x)
\]
\[
= d_{m+1} P_m(x) \quad \text{since } d_m(x) = D.
\]
This is a boundary, so \( (f-g)_x(x) = 0 \)
\[
equiv \quad f_x(x) = g_x(x), \quad \text{as } \quad H_x(f) = H_x(g).
\]
\( \square \)
Let $F : X \times I \to Y$ be a homotopy between $f$ and $g$. We will use to construct a chain homotopy $F$.

$C_m(X)$ is the free abelian graded by maps

$\sigma : \Delta^n \to X$

$\Delta^n \times \{ 0 \} \to \Delta^n \times \{ 0 \} \to X \times I \to Y$

Let $\Delta^n \times \{ 0 \}$ have vertices $[v_0, v_1, \ldots, v_n]$ and $\Delta^n \times \{ 1 \}$ have vertices $[w_0, w_1, \ldots, w_n]$

Let $\Delta_{n+1}^n$ be an $(n+1)$-simplex with vertices $[v_0, \ldots, v_n, w_1, \ldots, w_n]$ for $0 \leq i \leq n$

Then $\Delta^n \times I = \bigcup_{0 \leq i \leq n} \Delta_{n+1}^n$.
A chain homotopy is a map

\[ C_n(X) \xrightarrow{\partial_n} C_{n+1}(Y) \]
\[ P_n(g) = \sum_{0 \leq i \leq n} (\text{Hom}(g \times I), \Delta_{n+1}^i, c_{n+1}(g)) \]

Each \( f(g \times I), \Delta_{n+1}^i \) is a singular \((n+1)\)-
complex in \( \mathcal{Y} \). Direct calculation shows that this is the desired chain
diagram by \( \mathcal{G}(f) \) and \( \mathcal{G}(g) \).

It follows that \( H_n(f) = H_n(g) \). QED

Recall there are 4 Eilenberg–MacLane
axioms :

Dimension, Homotopy, Exactness
and Excision.
For exactness we need to define relative homology.

Con to homotopy axiom: Homotopy equivalent spaces have the same homology.

Proof: We have $X \xrightarrow{f} Y$ such that $g \circ f = \text{identity on } X$ and $f \circ g = \text{identity on } H_\ast(Y)$.

$H_\ast(X) \xrightarrow{f_*} H_\ast(Y) \xrightarrow{g_*} H_\ast(X) \xrightarrow{f_*} H_\ast(Y)$

$\text{identity on } H_\ast(X)$
This shows \( f^* \) and \( g^* \) are 1-1 and """""""""""""" are onto. Hence \( f^* \) and \( g^* \) are isomorphisms. QED

Back to relative homology:

Let \( A \subset X \) be spaces. \( C(A) \subset C(X) \). We have a chain map induced by the inclusion \( i: A \to X \).

\( C(X, A) = C(X) / C(A) \), the relative singular chain complex.

Define \( H_*(X, A) = H_*(C(X, A)) \).
Thus we have a short exact sequence of chain complexes

\[
0 \to C(A) \overset{i}{\to} C(X) \overset{j}{\to} C(X, A) \to 0
\]

This leads to a long exact sequence

\[
\cdots \to H_n(A) \overset{i_*}{\to} H_n(X) \overset{j_*}{\to} H_n(X, A) \overset{\partial}{\to} H_{n-1}(A) \to \cdots
\]

This is the desired long exact sequence. It is natural, i.e.

\[
\begin{align*}
\text{given} & \quad \begin{array}{c}
A \\ \downarrow f \\
B \\
\end{array} \overset{i}{\to} \begin{array}{c}
X \\ \downarrow g \\
Y \\
\end{array} \to \begin{array}{c}
(X, A) \\ \downarrow h \\
(Y, B) \\
\end{array}
\end{align*}
\]
we get a commutative diagram

\[ \cdots \to H_n(A) \to H_n(X) \to H_n(X \cup A) \to \cdots \]

\[ \downarrow f_* \quad \downarrow f_* \quad \downarrow f_* \quad \downarrow f_* \]

\[ \cdots \to H_n(B) \to H_n(Y) \to H_n(Y \cup B) \to \cdots \]

\[ \cong A \subset B \subset C \subset X. \text{ There is a LES} \]

\[ \cdots \to H_n(B \cup A) \to H_n(X \cup A) \to H_n(X \cup B) \to \cdots \]

**Proof**

\[ 0 \to C(B) \to C(A) \to C(A \cup B) \to 0 \]

\[ 0 \to C(B) \to C(X) \to C(X \cup B) \to 0 \]

\[ \text{in } C(X, A) \to C(X, A) \]
exact rows + columns
Third column lead to our LEM QED