Toward a proof that the free gp \( F_2 \) has \( F_n \) as a sub-gp.

A graph is a top. space consisting of edges + vertices

\[
\begin{align*}
\text{n}(X) &= \# \text{ of vertices} \\
\text{e}(X) &= \# \text{ of edges}
\end{align*}
\]

\[ x(X) = \text{n}(X) - \text{e}(X) \]

Thm: If \( X \) is a path connected finite graph then its homotopy is determined by
\[ d = X(X), \text{ it is homotopy equivalent to one with a single vertex and } \]

1-d edges, \( X \cong S^1 \vee S^1 \vee \ldots \vee S^1 \)

\[ \text{1-d circles.} \]

\[ \text{d = 1 tree} \]

\[ \text{Thin } \check{\pi}_1(\tilde{X}) \xrightarrow{\phi} \pi_1(X) \text{ is a covering with } \tilde{X} \text{ and } X \text{ path connected, then } \]

\[ \pi_1(\tilde{X}) \text{ is iso via } \phi \text{ to a subgroup of } \pi_1(X). \]
\[ \chi(X) = 3 - 6 = -3 \]

\[ \chi(\tilde{X}) = 1 - 2 = -1 \]

\[ \pi_1(\tilde{X}) = \mathbb{F}_2 \]

4 generators map to:

\[ \alpha_1 \mapsto b \]
\[ \alpha_2 \mapsto a b \alpha^{-1} \]
\[ \alpha_3 \mapsto a^2 b \alpha^{-2} \]
\[ \alpha_4 \mapsto a^3 \]

Remark: If \( \tilde{X} \rightarrow X \) is a covering
of path connected finite graphs, then $X(\hat{X})$ is divisible by $X(x)$.

For a nice path connected space $X$, (with $\tilde{\pi}_1 X = G$) has a path covering $\tilde{X}$ with $\tilde{\pi}_1 \tilde{X} = 0$. There is an action of $G$ on $\tilde{X}$, i.e., for $g \in G$ there is a homeomorphism $\hat{g} : \tilde{X} \to \tilde{X}$ with $\hat{0}(gg') = \hat{g} \hat{g}'$.

2. $g(x) = x \iff g = e$. The action is free.

3. $\tilde{X}/G = \tilde{X}$ where $\tilde{X}/G$ is
The orbit space. Each $x \in X$ defines a subset $G \cdot \{x\}$, called its orbit. An orbit is an equivalence class $x \sim x'$ if $x' = g(x)$ for some $g \in G$.

Let $H$ be a subgroup of $G$. Then we have $\tilde{X} \sim X/H$, where each $\tilde{X} \sim X/G$ maps is a covering $\pi_1(\tilde{X}/H) = H$.

Any path-connected cover of $X$ is $\tilde{X}/H$ for some $H \subset G$. 
An analogy between coverings and Galois theory.

universal cover \( \tilde{X} \)

intermediate covering for \( H \subset G \)

nice path compact space with \( G = \pi_1(X) \)

\( K = \bar{K}_G \)

field extension \( L = \bar{K}_H \)

intermediate field \( H \subset G \)

\( \bar{K}_H = \text{fixed part of } H = \{ x \in \bar{K} : h(x) = x \text{ for } h \in H \} \)
Theorem 1.30 \( \pi_1(Y, x_0) \rightarrow \pi_1(Y) \rightarrow X \)

covering

\[ Y \times I \rightarrow Y \]

Suppose \( \pi_1(Y) = G \), \( \pi_1(X) = G \), and both spaces are path conn.

To define \( \hat{g}(x) \) for \( x \in X \) and \( g \in G \):

Choose a closed path in \( X \) based at \( p(x) \) corresponding to \( g \in G \):

\[ \pi_1(X, p(x)) \simeq G \]. Lift it to \( Y \).

We get a path from \( x \) to another
At, which we define to be \( g'(x) \). The Thom implies depends only on \( g \) and \( x \) and not on the path chosen. Varying the path in \( X \) by a homotopy will not change the end point of its lifting.

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