Goal: Given a chain complex \( C \) and an abelian group \( A \), describe \( H^*_C(C \otimes A) \) and \( H^* \text{Hom}(C, A) \) in terms of \( H^*_C(C) \) and \( A \).

Recall a projective resolution of an \( R \)-module \( M \) is a LES of \( R \)-modules
\[
0 \to M \to P_0 \to P_1 \to \cdots
\]
where each \( P_i \) is projective over \( R \).

For another \( R \)-module \( N \), we get a chain complex
$P_0 \otimes_R N \leftarrow P_1 \otimes_R N \leftarrow P_2 \otimes_R N \leftarrow \cdots$

and a cochain complex

$\text{Hom}_R(P_0, N) \rightarrow \text{Hom}_R(P_1, N) \rightarrow \cdots$

The homology/cohomology of these are independent of the choices of the $P_i$, and only depend on $M$ and $N$. They are called $\text{Tor}_*(M, N)$ and $\text{Ext}_R^*(M, N)$.

If $R$ is a PID (e.g. if $R = \mathbb{Z}$) then any module $M$ has a projective resolution of length 1.
0 \leq M_0 \leq P_i \leq 0

so \text{Tor}_i \text{ and } \text{Ext}_i \text{ vanish for } i \leq 1. \\
\text{and } \text{Tor}_0^R(\mathbb{M}, \mathbb{N}) = \text{Mor}_R(\mathbb{M}, \mathbb{N}) \text{, } \text{Tor}_0 = ? \\
\text{Ext}_0^R(\mathbb{M}, \mathbb{N}) = \text{Hom}_R(\mathbb{M}, \mathbb{N}), \text{ Ext}_0 = ?

Both \text{Tor} \text{ and } \text{Ext} \text{ are additive}:
\text{Tor}_i(\mathbb{M} \Theta \mathbb{M}'', \mathbb{N}) = \text{Tor}_i(\mathbb{M}', \mathbb{N}) \oplus \text{Tor}_i(\mathbb{M}'', \mathbb{N})
\text{Tor}_i(\mathbb{M}, \mathbb{N} \Theta \mathbb{N}'') = \text{Tor}_i(\mathbb{M}, \mathbb{N}') \oplus \text{Tor}_i(\mathbb{M}, \mathbb{N}'')

and similarly for \text{Ext}:

Other properties:
1) \text{Tor}_i(\mathbb{N}, \mathbb{M}) = \text{Tor}_i(\mathbb{M}, \mathbb{N})
Ten in commuters in both variables

\text{not true of 1st}
Some specific calculations for $R = \mathbb{Z}$.

1) If $M = \mathbb{Z}$ (or any free abelian group), then $Tor_1(\mathbb{Z}, M) = 0 = \text{Ext}^1(\mathbb{Z}, M)$, because there is a resolution of length 0:

$$0 \gets M \gets P_0 \gets 0$$

2) If $M = \mathbb{Z}/m$ for $m > 1$. There is a resolution of the form:

$$0 \gets \mathbb{Z}/m \gets \mathbb{Z} \gets \mathbb{Z}/m \gets 0$$

The chain complex $N = A$ (for an abelian group $N = A$).
\[ \begin{array}{ccc} P_0 \otimes A & \rightarrow & P_1 \otimes A \\
\downarrow & & \downarrow \\
A & \leftarrow & A \end{array} \]

\[ \text{Tor}_0 = A/mA = A \otimes \mathbb{Z}/m = \text{ker} \ m \]

\[ \text{Tor}_1 = \text{ker} \ m = \{ a \in A : m a = 0 \} \]

\[ \text{Tor}_1 (\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p \]

The cochain cox is

\[ \begin{array}{ccc} \text{Hom} (P_0, A) & \rightarrow & \text{Hom} (P_1, A) \\
\downarrow & & \downarrow \\
\text{Hom} (\mathbb{Z}, A) & \rightarrow & \text{Hom} (\mathbb{Z}, A) \\
A & \leftarrow & A \end{array} \]
$\text{Ext}^0(\mathbb{Z}/m, A) = \ker m = \{ a \in A : m a = 0 \}$

$= \text{Hom}(\mathbb{Z}/m, A) = \text{Tor}_1(\mathbb{Z}/m, A)$

$\text{Ext}^1(\mathbb{Z}/m, A) = \text{coker} \ m$

$= \mathbb{Z}/m \otimes A.$

Note: If $A$ is torsion free, then

$\text{Ext}^0(\mathbb{Z}/m, A) = 0$

$\text{Hom}(\mathbb{Z}/m, A)$

\[\text{Thm 3A.3}\] Let $C$ be a chain complex of free abelian groups. For an abelian group $A$, there is a natural SES

$0 \rightarrow H_n(C) \otimes A \rightarrow H_n(C \otimes A) \rightarrow \text{Tor}_1(H_{n+1}(C), A) \rightarrow 0$
(Note the $\text{Tor}_1$ is $\neq 0$ only if $A$ and $H_{n-1}(C)$ both have torsion.) The sequence splits, but not naturally.

**Thm 3.2** For $C$ and $A$ as above let

$$H^*(C; A) = H^*(\text{Hom}(C, A)).$$

Then there is a natural SES

$$0 \rightarrow \text{Ext}^1(H_{n-1}(C), A) \rightarrow H^n(C; A) \rightarrow H^0(\text{Hom}(H_n(C), A)) \rightarrow 0$$

It is evident but not naturally.

**Proof of 3.2** We have

$$0 \rightarrow \mathbb{Z}_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow \text{im} A_1 \rightarrow \mathbb{Z}_n \rightarrow \cdots \rightarrow C_1 \rightarrow 0$$

Diagram 1:

$$0 \rightarrow \mathbb{Z}_{n-1} \rightarrow C_{n-1} \rightarrow B_{n-2} \rightarrow 0$$
This means we have a SES of chain complexes
\[ 0 \to \mathbb{Z} \to C \to B \to 0 \]
where \( \mathbb{Z} \) and \( B \) have trivial boundary operators. The LES in \( H_* \) reads

\[
\cdots \to H_{n+1}(B) \to H_n(B) \to H_n(C) \to H_n(B) \to H_{n-1}(B) \to \cdots
\]

\[
B_n \to \mathbb{Z}_n \to H_n(C) \to B_{n-1} \to \mathbb{Z}_{n-1} \to H_{n-1}(C) \to \cdots
\]

Since each row of \( \mathbb{Z} \) is split, tensoring with \( A \) preserves exactness. Note that

\[ 0 \to B_n \xrightarrow{\text{in}} \mathbb{Z}_n \to H_n(C) \to 0 \]

is a free resolution of \( H_n(C) \).
OOPS Crucial property of Tor and Ext

A SES \( 0 \to N' \to N \to N'' \to 0 \)
leads to a LES in Tor and Ext

\[
\text{Tor}_n^i(M, N) \to \text{Tor}^{i+1}_n(M, N) \to \text{Tor}^{i+1}_n(M, N''') \to \text{Tor}^{i+1}_n(M, N') \to \ldots
\]

\[
\ldots \to \text{Ext}^i(M, N) \to \ldots
\]

Why? Tensorsing a SES with \( X \) does not preserve exactness in general.

But it does if

1) SES is split

2) \( X \) is free or projective.