Let $C'$ and $C''$ be chain complexes of free abelian $\mathbb{Z}$-modules and let $C = C' \oplus C''$.

**Künneth Theorem**

There is a natural splitting

$$0 \to \bigoplus H_i(C') \otimes H_{n-i}(C'') \to H_n(C) \to \bigoplus \text{Tor}_i^\mathbb{Z}(H_i(C'), H_{n-i}(C'')) \to 0$$

**Proof** is similar to the previous ones.

**Remark**

Let $C = \mathbb{Z} e_0 \oplus \mathbb{Z} e_2 \oplus \mathbb{Z} e_0 \oplus \mathbb{Z}$.

Its homology is that of $\mathbb{R}P^3 = SO(3)$.

$$\text{Hom}(C, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

Note that this is the "mirror" of $C$. 


For any closed (compact without boundary) n-manifold \( M \) there is a chain \( C \)
\[ C = C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_n \]
with \( H_x(\mathbb{Z}) \cong H_x(M) \).

Let \( C^i = \text{Hom}(C_i, \mathbb{Z}) \), so we get a cochain complex
\[
\text{Hom}(\mathbb{Z}; C) : C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow \cdots \rightarrow C^n
\]

\[ C^i \cong C_{n-i} \quad \text{Poincaré duality} \]

\[ \text{Hom}(\mathbb{Z}; \mathbb{Z}) : H^i(M; \mathbb{Z}) = H^{n-i}(M) \]

We also know \( H^i(M; \mathbb{Z}) \) is related to \( H^i(M) \) and \( H_{i-1}(M) \).
Example of Kunneth Theorem

Let \( C' \cong C'' \cong \mathbb{Z}/2 \mathbb{Z} \)

(associated with \( \mathbb{RP}^2 \))

\[ H_n(C') = H_n(C'') = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 2/2 & \text{for } n = 1 \\ 0 & \text{for } n = 2 \end{cases} \]

\[ C = C' \otimes C'' \]

\[ C_m = \bigoplus_{0 \leq i \leq m} C_i \otimes C_{m-n} = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 2 \\ \mathbb{Z} & \text{for } n = 3 \\ \mathbb{Z} & \text{for } n = 4 \end{cases} \]

For \( x \in C'_n \) and \( y \in C''_m \),

\[ d_i : (x \otimes y) = d_i(x) \otimes y + (-1)^i x \otimes d_i''(y) \]

\[ \forall \alpha \in d_i(x) \otimes y \]

\[ d(\alpha \otimes y) = 0 = d(\beta \otimes \alpha) \]
\[ d_2(\alpha \otimes x) = d_0(\alpha) \otimes x + \alpha \otimes d_2(x) = 2 \alpha \otimes B \]
\[ d_2(G \otimes B) = 0 \]
\[ d_2(\chi \otimes x) = 2 \beta \otimes x \]
\[ d_3(G \otimes x) = d_1(G) \otimes x - \beta \otimes d_2(x) = -2 \beta \otimes B \]
\[ d_3(\chi \otimes x) = d_2(\chi) \otimes B + \chi \otimes d_2(B) = 2 \beta \otimes B \]
\[ d_4(\chi \otimes x) = d_2(\chi) \otimes \chi + \chi \otimes d_2(x) = 2 \beta \otimes \chi + 2 \chi \otimes B \]

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 & 4 \\
2 \leq & d_1 \leq & d_2 \leq & d_3 \leq & d_4 \leq \\
\end{array}
\]

\[
\begin{bmatrix}
2 \beta \\
0 \\
\end{bmatrix} \leftarrow \begin{bmatrix}
\chi \otimes X \\
\alpha \otimes X \\
\end{bmatrix} \leftarrow \begin{bmatrix}
2 \beta \otimes \chi \\
2 \beta \otimes \chi \\
\end{bmatrix} \leftarrow \begin{bmatrix}
\beta \otimes X \\
\chi \otimes \beta \\
\end{bmatrix}
\]

\[
H_0(C) \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
2 & 2 \beta \otimes x & 2 \chi & 2 \beta & 0 \\
\end{pmatrix}
\]
For space $X$ and $Y$, there is a map
\[ s(x) \otimes s(y) \rightarrow s(X \times Y) \]
which is chain homotopy equivalent, so
\[ H_*(X \times Y) = H_* (S(X \times Y)) = H_* (s(x) \otimes s(y)) \]
which is related to $H_* X$ and $H_* Y$ by the Künneth Theorem. Our example computes $H_* (RP^2 \times RP^2)$.
Recall the definition of CW-\(cx\) (see notes of March 4).

Let \(X\) and \(Y\) be CW-complexes. What about \(\text{Map}(X,Y)\), the space of continuous maps \(X \to Y\)?

Plan (Milošević).

\(\text{Map}(X,Y)\) is homotopy equivalent to a CW \(cx\).

Example: \(X = S^1\), \(Y = S^{n+1}\).

\(\text{Map}_*(X,Y)\) = base point preserving maps.

\[\approx \Sigma^n \Omega^{n+1} \vee \ldots\]