CW complexes

\[ X^0 \subset X^1 \subset X^2 \subset \cdots \subset X \]

\[ X^0 = \text{discrete} \quad X^k = \text{\(k\)-skeleton of \(X\)} \]

\[ X^n \text{ is obtained from } X^{n-1} \text{ by attaching} \]

some \(n\)-cells, i.e., copies \(D^n\) glued or

by attaching maps \(S^{n-1} \times X^{n-1}\)

The cellular chain \(C_\bullet(X)\) of \(X\) is:

\[ C_n(X) = \text{free abelian \( \mathbb{Z} \) on the} \]

\(n\)-cells of \(X\)

To define \(d\), note that
\[
\frac{X^{n-1}}{X^{n-2}} = \bigvee_{\alpha \in \Sigma^{n-1}}
\]

\[\pi_{n-1}(X^{n-1}/X^{n-2}) = C_{n-1}(X)\]

is free abelian gp on the \((n-1)\)-cells

Each generator of \(C_n(X)\) is associated with an \(n\)-cell and an attaching map

\[S^{n-1} \xrightarrow{\varepsilon} X^{n-1} \xrightarrow{\varphi} X^{n-1}/X^{n-2}\]

which defines an elt in \(\pi_{n-1}(X^{n-1}/X^{n-2}) = C_{n-1}\). This gives us a chain complex.

Nice feature:

If \(X\) and \(Y\) are CW-complexes, then \(X \times Y\) can be given a CW-structure.
where
\[(X \times Y)^n = \bigcup_{0 \leq i \leq n} X^i \times Y^{n-i}\]
and \(C_* (X \times Y) = C_* (X) \otimes C_* (Y)\).

We can show \(H_* (X) = H_* (C_* (X))\).

\(C_* (X)\) is much easier to deal with than the singular chain \(\mathbb{S}_* (X)\).

Example: \(\mathbb{C}P^n\). This has a CW-structure with one cell in each even dimension from 0 to 2n.

\(\mathbb{C}P^0 = \mathbb{C}\).
\(\mathbb{C}P^1 \cong S^2\).
\(\mathbb{C}P^n = \text{space of complex lines through } 0 \text{ in } \mathbb{C}^{n+1}\).
A point in $\mathbb{C}P^n$ can be described by
$$[x_0, x_1, \ldots, x_n], \quad x_i \in \mathbb{C} \text{ not all 0}\)$$

corresponding to the line in $\mathbb{C}^{n+1}$ through $0$ and $(x_0, \ldots, x_n)$
$$[x_0, x_1, \ldots, x_n] = [\lambda x_0, \lambda x_1, \ldots, \lambda x_n]$$

for $\lambda \in \mathbb{C} \setminus \{0\}$

$\mathbb{C}P^n = \{[x] \in \mathbb{C}P^n \text{ with } x_n = 0\}$

and $\mathbb{C}P^n = \mathbb{C}P^{n+1} = \{[x] : x_n \neq 0\}$

$= \{[x_0, \ldots, x_{n-1}, 1]\} = \mathbb{C}^{n-1} = \text{int} \mathbb{D}^{n-1}$

$\mathbb{C}P^n$ can be obtained from $\mathbb{C}P^{n-1}$ by

attaching $2n$-cells. The attaching map
\[ C^n \to S^{2n-1} \to \mathbb{CP}^{n-1} \]

\[ (x_0, \ldots, x_{n-1}) \mapsto [x_0, \ldots, x_{n-1}] \]

\( C_x(\mathbb{CP}^n) \) has trivial boundary operator

\[ H_i(\mathbb{CP}^n) = \begin{cases} 
\mathbb{Z} & \text{if } i \text{ is even and } \\
0 & \text{otherwise} 
\end{cases} \]

2) \( X = \mathbb{RP}^n \). Similar discussion leads to one cell in each dimension from 0 to \( n \).

\[ d_i = \begin{cases} 
2 & \text{if } i \text{ is even and } \\
0 & \text{otherwise} 
\end{cases} \]

for \( 1 \leq i \leq n \)

\[ z^0, z^2, z^0, z \]

\[ \mathbb{RP}^2 \to \mathbb{RP}^3 \]
$H_2 \mathbb{RP}^n = \begin{cases} 2 & i = 0 \\ \mathbb{Z} & i \text{ is even and } 0 < i < n \\ 0 & i = n \text{ and } n \text{ is odd} \\ \mathbb{Z} & \text{otherwise} \end{cases}$

How to compute $d_i$:

\[
\begin{array}{c}
\mathbb{Z}^{i-1} \rightarrow \mathbb{RP}^{i-1} \rightarrow \mathbb{RP}^i / \mathbb{RP}^{i-2} = \mathbb{Z}^{i-1} \\
\end{array}
\]

We need to find its degree.

Related fact: The antipodal map

\[
\sigma^n : S^n \rightarrow S^n \text{ has degree } (1)^{n+1} \\
\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \text{ is multiplied by the matrix } -I
\]

\[
\det (-I) = (-1)^{n+1}
\]

\[
\deg q_i = 1 + (-1)^i = 1 + \text{degree of antipodal on } S^{i-1}
\]
New topic: Cup products.

Recall there is a map
\[ H^*_x(X) \otimes H^*_y(Y) \rightarrow H^*_{x+y}(X \times Y) \]

There is also a map
\[ H^*_x(X) \otimes H^*_y(Y) \rightarrow H^*_{x+y}(X \times Y) \]

e.g., \[ H^*_x(X) \otimes H^*_y(X) \rightarrow H^*_{x+y}(X \times X) \]

This makes \( H^*_x(X) \) a graded ring.

i.e., given \( \alpha \in H^*_x(X) \) and \( \beta \in H^*_y(X) \)

we have \( \alpha \cup \beta \in H^*_{x+y}(X) \), the
cup product of $\alpha$ and $\beta$:

Formal properties:

1) **Natural** $X \to Y$
   
   induces a ring hom $H^*(X) \to H^*(Y)$

2) **Associativity**

3) **Distributive law**

4) $\beta \alpha = (-1)^{\text{dim} \alpha \times \text{dim} \beta}$

odd dimensional elements anti-commute

even dimensional elements commute with everything

Geometric property
Let $M$ be a closed oriented $n$-manifold, e.g. $S^1 \times S^1$.

Poincaré Duality says $H^i(M) \cong H_{n-i}(M)$

$$H^i(M) \otimes H^j(M) \rightarrow H^{i+j}(M)$$

$$H_{n-i}(M) \otimes H_{n-j}(M) \text{ intersection} \rightarrow H_{n-i-j}(M)$$

Problem: In a 12-hour period, say 6 AM to 6 PM. How many times do the minute + hour hands point in the same direction?