Recall $\dim \mathbb{C}P^{\infty} = 2$, so $H^* (\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z} [x]$ with $|x| = 2$

$H^* (\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2 [x]$ with $|x| = 1$

Each space has a simple nontrivial homotopy group $\pi_i (\mathbb{R}P^\infty) = \mathbb{Z}/2$ for $i = 1$

$\pi_2 (\mathbb{C}P^\infty) = \mathbb{Z}$

Recall $\mathbb{R}P^n$ is double covered by $S^n$.

This means their homotopy groups agree above $\dim 1$. $S^n$ is $(n+1)$-connected so

$\pi_i (\mathbb{R}P^n) = \begin{cases} \mathbb{Z}/2 & \text{for } i = 1 \\ 0 & \text{for } 1 < i < n \end{cases}$
Let $n \to \infty$ and get $\pi_i \mathbb{R}P^{\infty} = S^{2i/2}$.

There is a fiber sequence

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n$$

unit vectors on lines through $0$ in $\mathbb{C}^{n+1}$

Hence there is a LES

$$\cdots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{C}P^n) \to \pi_{k-1}(S^1) \to \cdots$$

\[\begin{align*}
    \pi_k S^1 &= S^2 & \text{for } k \geq 1 \\
    \pi_k S^1 &= 0 & \text{for } k \geq 0
\end{align*}\]

and \(\pi_k(S^{2n+1}) = 0\) for \(k < 2n+1\)

\[\begin{align*}
    \pi_k \mathbb{C}P^n &= S^2 & \text{for } k = 2 \\
    \pi_k \mathbb{C}P^n &= 0 & \text{for } k = 1 \\
    \pi_k \mathbb{C}P^n &= 0 & 2 < k < 2n+1
\end{align*}\]
Let \( n \to \infty \) and get \( \cap_k \mathbb{C}P^\infty = \begin{cases} \mathbb{C} & \text{if } k = 2 \\ \emptyset & \text{if } k \neq 2 \end{cases} 

For an abelian \( A \) and an integer \( n > 0 \), there is a unique (up to isy equiv) space \( K(A, n) \) with

\[
\pi_i K(A, n) = \begin{cases} 0 & \text{for } i = n \\ 0 & \text{for } i \neq n \\ \end{cases}
\]

Eilenberg–Mac Lane space

\[
\mathbb{S}^k = K(\mathbb{Z}, 1) \quad \mathbb{C}P^\infty = K(\mathbb{Z}, 2) 
\]

\[
\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1) 
\]

Mİaux classified space construction.
Def: For spaces $X$ and $Y$, their join $X \star Y = \Sigma \times \Delta [x] \times Y \cup \{ (x, 0, y) \sim (x, 0, y) \}$

\[
\begin{align*}
S^m \times S^n & = S^{m+n+1} \\
S^0 \times S^0 & = S^1
\end{align*}
\]

Proof: If $X$ is $(m-1)$-connected and
$Y$ is $(n-1)$-connected, then $X \times Y$ is $(m+n)$-connected.
Let $G$ be a topological group.

$$E_n G = \underbrace{G \times G \times \cdots \times G}_{n+1 \text{ factors}}$$

It is $(n-1)$-connected.

We have inclusions $E_n G \hookrightarrow E_{n+1} G \hookrightarrow \cdots$.

Let $EG = \lim_{\to} E_n G$. It is contractible.

$G$ acts on $E_n G = G^{n+1} \times I^n / I^n$

via

$$g \left( (g_0, g_1, \ldots, g_n) \times_t \ldots \times_t \right)$$

$$= (g_0 \circ g_1, \ldots, g_0 \circ g_1) \times_t \ldots \times_t$$

This action is free.

$B_n G = \text{orbit space } E_n G / G$

There is a fiber bundle.
\[ G_{1} \to E_{n}G_{n} \to B_{n}G_{n} \]

Hence there is a LES up to \( n \).

Let \( n \to \infty \) \( BG_{i} = \lim_{n \to \infty} B_{n}G_{i} \) = classifying space of \( G_{i} \).

\[ \pi_{i+1}G_{i} = \pi_{i+1}BG_{i} \quad \text{for all } i \geq 0 \]

If \( G_{i} \) is discrete then \( \pi_{0}G_{i} = G_{i} \).

So \( \pi_{1}BG_{i} = \begin{cases} G_{i} & \text{for } i = 1 \\ 0 & \text{for } i > 1 \end{cases} \)

Examples:

1) \( G_{1} = C_{2} = S^{0} \) \( E_{n}G_{1} = S^{n} \) with antipodal group action

\( B_{n}G_{1} = S^{n}/C_{2} = \mathbb{RP}^{n} \)

2) \( G_{2} = S^{1} \) with usual topology
$E_n G = S^{2n+1}$ and $B_n G = \mathbb{C}P^n$.

3) $G = \mathbb{Z}$ with discrete topology

$E_n G = ?$, $B_n G = ?$

$\lim_{n \to \infty} B_n G \cong S^1$

4) $G = C_p \hookrightarrow S^1$

$E_n C_p \hookrightarrow E_n S^1 = S^{2n+1} \subset C^{-n+1}$

 $(n-1)$-connected subspace $S^{2n+1} \subset S^1$

$C_p$ acts freely on both $E_n C_p$ and $E_n S^1$

so we have a map

$B_n C_p \to S^{2n+1}/C_p$

One can show

This map induces

$\bar{C}_i$ in $\check{H}_i$, for

$i < n$. 
\[
H_n \left( \mathbb{Z}^{2n+1} \mathbb{Z} \right) = \begin{cases} 
\mathbb{Z} & \text{for } i = 0 \text{ and } 2n+1 \\
\mathbb{Z}/p & \text{for } i \text{ odd and } 0 < i < 2n+1 \\
0 & \text{otherwise}
\end{cases}
\]

It has a cellular chain \( CX \)
\[
\mathbb{Z} \mathbb{C} \mathbb{C} \cdots
\]

Thus we know \( H_x BG \)

Nice property for discrete \( G \)

Suppose \( G \) acts freely on a space \( X \). Then we get a covering \( \tilde{X} \to X/\Gamma \)

Then for \( X \) paracompact, there is a unique (up to homotopy) map

\( \tilde{X} \to X/\Gamma \)
The construction of $BG$ is functional on $G$, a gp hom $G \to H$ induces a map $BG \to BH$.

Let $G$ be abelian and consider the multiplication map $G \times G \to G$. It is a hom; we have $B(G \times G) \to BG$. 
$BG \times BG$

This makes $BG$ itself a topological group.

Example: $G = S^1$

$BG = CP^\infty$

The map $CP^\infty \times CP^\infty \rightarrow CP^\infty$ is defined as follows:

$[\ldots, z_1, z_0] \in CP^\infty$ where $z_0 = 0$

let $f(t) = \sum z_n t^n \in C[t]/C^x$

the $z_n$ cannot all be 0.
Let \([z_0', z_1', \ldots] \) correspond to \(q(t)\)

Let \(h(t) = f(t)g(t)\). This defines

an map \(CP^\infty \times CP^\infty \rightarrow CP^\infty\)

This makes \(CP^\infty\) an abelian monoid. It can be converted into a gp without changing the top. type. HANDWAVING

This means \(CP^\infty = K(\mathbb{Z}, 2)\) has a classifying space \(K(\mathbb{Z}, 3)\).

For any abelian gp \(A\), \(BA\)
is also a topological abelian group, so we can iterate and form $B^k A$.

If $A$ is a discrete abelian group, then $B^n A = K(A, n)$.

**Question:** What is $H^*(BA)$?

**Thm:** For a discrete group $G$,

$$H^*(BG; \mathbb{Z}) = \text{Ext}^n_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$$

Here $\mathbb{Z}[G]$ is the integer group ring of $G$, i.e., free abelian group on the set $G$. 
with multiplication defined by

\[ [g_1] [g_2] = [g_1 g_2] \quad \text{for } g_1, g_2 \in G \]

The \( \mathbb{Z}[G] \)-module structure on \( \mathbb{Z} \)
is defined by a ring homomorphism

\[ \mathbb{Z}[G] \to \mathbb{Z} \]

\[ [g] \mapsto 1 \]

Example: \( G = C_2 = \{ e, x \} \) with \( x^2 = e \)

In the ring \( \mathbb{Z}[G] \), the element \([e]\) is the unit, so we will write \([e] = 1\)

\([x]^2 = 1 \quad [x] - 1 \]

Let \( \chi = [x] - [e] \)
\[ x^2 = (\sum a_x - \sum e)^2 = (a_x^2) - 2(a_x \sum e) + (\sum e) \]
\[ = (\sum e) - 2(a_x \sum e) + 2(\sum e) \]
\[ = 2(\sum e) - 2(a_x \sum e) = 2x \]

This means \( z \sum f \times \sum f \sum (\frac{1}{(x^2 - 2x}) \)

On which \( \exists \sum f \xrightarrow{\epsilon} \sum \)

\( \mathbb{R}^n \times I \xrightarrow{\pi} \mathbb{R} \)

We need a free \( \mathbb{R} \)-resolution of \( \mathbb{Z} \)

Think about this for next time.