Recall a dense class $C$ of abelian groups is a collection closed under extensions, i.e. given a SES

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

if $A'$ and $A''$ are in $C$ then $A$ is in $C$.

Examples

1. Torsion gps
2. Torsion gps with no elements of order $p$. 
3. $p$-torsion gps.
A hom $A \rightarrow B$ is a $C$-iso if key $Y$ and cok key $Y$ are in $C$.

A map of spaces $x \rightarrow y$ is a $C$-equivalence if it induces a $C$-iso in $H_i$ for $i > 0$.

If $C$ is the class of torsion gps then a $C$-equiv. is the same as a rational equiv. i.e. a map inducing an iso in $H_i \cap (\sim \mathbb{Q})$.

If $C$ is as in (2) then a $C$-equiv. is the same as a $p$-local equiv.
i.e. a map inducing an iso in $H_x (\mathbb{Z} / 2 \mathbb{Z})$.

We want to find $\pi_x S^n$ modulo torsion, i.e. find $\pi_x S^n \otimes \mathbb{Q}$. We will use Serre's method.

We know

$$H^* (k(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} \mathbb{Q}[x_n] & \text{for } n \text{ even} \\ \mathbb{Q}[x_n] / (x_n^2) & \text{for } n \text{ odd} \end{cases}$$

where $x_n \in H^n$.
For $n$ odd, the map $S^n \to K(2,m)$ inducing an iso in $\pi_n$ is a rational equivalence. This implies $\pi_i(S^n) \otimes \mathbb{Q} = \pi_i K(2,m) \otimes \mathbb{Q} = \{0\}$ for $i < n$.

For $i > n$, $\pi_i S^n$ is torsion. It is also finite.

For even $n$, the map $S^n \to K(2,m)$ is not a rational equivalence.
Recall for $n = 2$, the fibers of this map in $S^3$. This means
\[ \pi_i S^2 = \begin{cases} S^2 & \text{for } i = 2 \\ \pi_i S^3 & \text{for } i \geq 2 \end{cases} \]

\[ \pi_i S^2 \otimes \mathbb{Q} = \begin{cases} 0 & \text{for } i = 2 \text{ or } 3 \\ \mathbb{Q} & \text{otherwise} \end{cases} \]

\[ \pi_i S^2 = \begin{cases} 2 & \text{for } i = 2 \\ 2 & \text{finite } i = 3 \\ \text{finite } i \geq 3 \end{cases} \]

We had the fibre $S^2$ from $X \to S^2 \to \mathbb{C}P^\infty$.
Now consider

\[ X_4 \rightarrow \mathfrak{S}_4 \rightarrow K(2, 3, 4) \]

and use rational coefficients.
\( y \in \mathbb{N}^7 \times \mathbb{N} \quad x \in \mathbb{N}^8 \times \mathbb{N} \\
\alpha_8(y) = x^2 \\
\alpha_8(x^n y) = x^{n+2} \)

\( \implies H^* \left( K(2,4), \mathbb{Q} \right) \)

\text{Hence} \quad H_n \left( X_4, \mathbb{Q} \right) = \begin{cases} 
\mathbb{Q} & \text{for } i = 0 \\
0 \text{ else} 
\end{cases}

\text{Hence} \quad \pi_n \left( X_4 \right) = \begin{cases} 
\mathbb{Z} & \text{for } i = 4 \\
\mathbb{Z} \oplus \text{finite} & \text{for } i = 7 \\
\text{finite} & \text{other } i \geq 4 
\end{cases}
Similarly, \( T^i_s \mathbb{S}^{2n} = \begin{cases} \infty & \text{for } i = 2n, \\ \text{finite} & \text{for other } i > 2n. \end{cases} \)

Thus, \( S^{2n} \) has an element of infinite order. How to construct it?

Consider \( S^{2n} \times S^{2n} \) as a CW-complex. It has 2 cells in dimension \( 2n \) and 1 in \( 4n \). Its 2n-skeleton is \( S^{2n} \cup S^{2n} \). We have an attaching map.
Claim \( w \) has infinite order. Consider the CW complex formed by using \( w \) as an attaching map.

\[
\begin{array}{cccc}
S^{4n-1} & \xrightarrow{f} & S^{2n} \cup S^{2n} & \text{for the top cell.} \\
S^{2n} & \xrightarrow{w} & S^{2n} & \text{fold}
\end{array}
\]
Consider $H_\ast(S^2 \times S^2) = \bigoplus_{i=0}^{2n} \bigoplus_{j=\geq 2n} \mathbb{Z}$ for $i = 0, 2n, 4n$ and $ab$

\[ H^1 W = \bigoplus_{i=0}^{2n} \bigoplus_{j=\geq 2n} \mathbb{Z} \]

Claim $x^2 = 2y \in H^{4n} W$

$g^*(x) = ax + b$

$g^*(y) = ab$

$g^*(x^2) = (a + b)x = a^2 + 2ab + b^2 = 2ab$

$= g^*(2y)$

Hence $x^2 = 2y$ and $H^{4n}(g)$ is an isomorphism.

To show $w \in H_{4n-1} S^{2n}$ has infinite
Subtle question: Do there a map
\[ S^{n-1} \rightarrow S^{2n} \rightarrow W' \]
\[ \delta \]
where \( x = y' = \text{gen of } H^n W' \)?

i.e., is \( W \in \Omega_{n-1} S^{2n} \) divisible by \( 2 \)?

Answer (Hurewicz Theorem due to Adams)
only if \( n = 1, 2 \) or 4. In those cases \( y' \) is a Huref

\[ n = 1 \quad S^3 \rightarrow S^2 \]
\[ n = 2 \quad S^7 \rightarrow S^2 \]
\[ n = 4 \quad S^{15} \rightarrow S^8 \]
In each case \( \chi^2 = \text{gen of } H^+ \text{ (making core) } \)